

A Useful Integral for Wireless Communication Theory and Its Application in Amplify-and-Forward Multihop Relaying

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Abstract—In this paper, infinite integrals involving the product of Bessel functions of different arguments are solved in closed-form. These solutions provide novel expressions in the error rate analysis for wireless amplify and forward (AF) systems with an arbitrary number of variable-gain relays. Here we show that the error probability can be described by generalized hypergeometric functions, namely Lauricella functions. This work represents a significant improvement on previous contributions and extends previous formulas pertaining to dual-hop transmissions over identical Nakagami- m fading channels. Numerical examples show an excellent match between simulation and theoretical results.

I. INTRODUCTION

Recently, the multi-hop concept has gained momentum in the context of cooperative wireless systems where relaying is used as a form of spatial diversity to overcome highly shadowed or deeply faded links [1]. The main idea is that communication is achieved by relaying the signal from the source to the destination via many intermittent terminals in between called relays. With relays that merely amplify and forward the incoming signal prior to relaying, AF transmission is the simplest and the cheapest to implement. Performance of such a system can be analyzed through the theoretical evaluation of certain performance metrics, namely, the average error probability. Closed-form analysis of such performance measure provides useful insights for comparative evaluation and optimization of parameters and protocols. So far, despite many valuable contributions [2]-[8], the dual-hop case is still incomplete and there are no closed-form expressions for AF multihop systems with an arbitrary number of relays. In [2], [3], Hasna and Alouini presented an MGF-based error probability analysis for dual-hop relaying system over identical Nakagami fading. Only recently have the authors in [4] considered the non-identical case, but merely for integer values of the Nakagami fading parameter. Certain results for more complicated systems, namely, multi-hop [5]-[8] are also available. In [5], [6], lower bounds on the bit error probabilities of AF multihop transmission systems with variable-gain relays and fixed-gain relays over Nakagami- m and generalized fading channels were derived utilizing the statistics of the geometric means of the individual link signal to noise ratios (SNRs). However, This lower bound significantly loses its tightness

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with increasing SNR. The most valuable contributions concerning the exact evaluation of the error probability can be found in [7] and [8]. In reference [7], the error probabilities are expressed as a double infinite integrals expressed in terms of the moment generating function (MGF) of the instantaneous received SNR. In the theoretical approach presented in [8], the error probability performance of an AF multihop system is evaluated using single-integral expressions obtained in terms of the MGF of the reciprocal of the instantaneous received SNR. Our approach is inspired by [8] and generalizes both [2] and [4].

This paper provides a unified error analysis of multi-hop AF for independent but non-identical Nakagami- m fading links. It turns out that the average error probability belongs to a special class of generalized hypergeometric series. These are the Lauricella's multivariate hypergeometric functions [9] of N variables $F_C^{(N)}$ for which some quite substantial mathematical apparatus is already known, like convergence properties and some analytical continuation formulas. Although the results are not expressible in common simple functions, they are at least expressible in this known type of functions, a significant improvement over previous results. For a dual-hop AF transmission over non identical Nakagami- m fading, the obtained formula involves Appell's hypergeometric [9] and Meijer's-G [10] functions.

The rest of this paper is organized as follows. Section II derives closed-form solutions to the infinite integral containing the product of Bessel functions. In section III, the error-rate performance for a variety of modulation schemes of AF multihop relaying systems with variable-gain relays is evaluated over independent but not necessarily identically distributed Nakagami- m fading channels. Some numerical results are provided in section IV. Finally, we conclude the paper while summarizing the main results in section V.

II. SOLUTION TO THE INFINITE INTEGRAL

This paper first addresses the calculation of the integrals

$$I(\nu, \mu, a, \Lambda, \beta) = \int_0^\infty s^\nu J_\mu(a\sqrt{s}) \prod_{i=1}^N K_{\lambda_i}(b_i\sqrt{s}) ds, \quad (1)$$

where

$$\begin{cases} \Lambda = \{\lambda_1, \dots, \lambda_N\}, \\ \beta = \{b_1, \dots, b_N\}, \\ \Re(a), \mu > 0, \\ N > 1, \end{cases} \quad (2)$$

$J_\mu(\cdot)$ is the bessel function of the first kind and order μ [10, Eq. 8.440] and $K_\lambda(\cdot)$ is the modified bessel function of the second kind and order λ [10, Eq. 8.485]. The integral in (1) occurs in a number of wireless applications including the evaluation of the error probabilities and the ergodic capacity of wireless multihop systems. Yet, to the best of the authors knowledge, a closed-form solution for this integral is not known. Furthermore, a closed-form solution to the special case of (1) obtained when $N = 2$, that is

$$I_s(\nu, \mu, a, \lambda_1, \lambda_2, b_1, b_2) = \int_0^\infty s^\nu J_\mu(a\sqrt{s}) K_{\lambda_1}(b_1\sqrt{s}) K_{\lambda_2}(b_2\sqrt{s}) ds \quad (3)$$

is not widely known and seems to have been found only when $b_1 = b_2$ and $\lambda_1 = \lambda_2$ see [11]. None of references [11] or [10] gives a closed-form solution for either I or I_s . Nor does Mathematica give a closed-form solution for I or I_s . In this paper, we derive an explicit and general solution to (1) for any number $N > 1$. Our analysis is only valid for real-valued non-integer λ_i . Nevertheless, practically, very similar results can be obtained at λ_i and $\lambda + \epsilon$ for sufficiently small ϵ values. By expressing the Bessel functions in terms of hypergeometric functions, namely, using

$$K_\lambda(z) = 2^{\lambda-1} \Gamma(\lambda) z^{-\lambda} {}_0F_1(; 1 - \lambda, \frac{z^2}{4}) + 2^{-\lambda-1} \Gamma(-\lambda) z^\lambda {}_0F_1(; 1 + \lambda, \frac{z^2}{4}), \quad (4)$$

and

$$J_\mu(z) = \frac{1}{\Gamma(\mu+1)} \left(\frac{z}{2}\right)^\mu {}_0F_1\left(; 1 + \mu, -\frac{z^2}{4}\right), \quad (5)$$

where ${}_0F_1(a, b, z)$ denotes the confluent hypergeometric function [10], an alternative expression for I is shown to be given by

$$I(\nu, \mu, a, \Lambda, \beta) = \frac{a^\mu}{2^\mu \Gamma(\mu+1)} \int_0^\infty s^{\nu+\frac{\mu}{2}} K_{\lambda_N}(b_N\sqrt{s}) {}_0F_1\left(; 1 + \mu, -\frac{a^2 s}{4}\right) \prod_{i=1}^{N-1} [V_i + W_i] ds, \quad (6)$$

where

$$V_i = 2^{\lambda_i-1} \Gamma(\lambda_i) (b_i\sqrt{s})^{-\lambda_i} {}_0F_1\left(; 1 - \lambda_i, \frac{b_i^2 s}{4}\right), \quad (7)$$

and

$$W_i = 2^{-\lambda_i-1} \Gamma(-\lambda_i) (b_i\sqrt{s})^{\lambda_i} {}_0F_1\left(; 1 + \lambda_i, \frac{b_i^2 s}{4}\right). \quad (8)$$

In subsequent derivations, a more convenient expression for the product involved in (6) will be given using the following lemma.

Let V_1, \dots, V_N and W_1, \dots, W_N denote two sets of N variables. Then, the following equality holds

$$\prod_{i=1}^{N-1} (V_i + W_i) = \sum_{i=0}^{N-1} \sum_{\tau(i, N-1)} \prod_{k=1}^{N-1} V_k^{i_k} W_k^{1-i_k}, \quad (9)$$

where $\tau(i, N-1)$ is the set of $N-1$ -tuples such that $\tau(i, N-1) = \{(i_1, \dots, i_{N-1}) : i_k \in \{0, 1\}, \sum_{k=1}^{N-1} i_k = i\}$. Indeed, by expanding the left side of (9), we can clearly notice that the i -th term can be viewed as $\binom{N-1}{i}$ combinations of the product

of i_k variables V_k and $1 - i_k$ variables W_k . Note that, when V_k and W_k are equal, (9) reduces to the Newton's binomial. Using the above equality, (6) will be given by

$$I(\nu, \mu, a, \Lambda, \beta) = \frac{a^\mu \prod_{k=1}^{N-1} \left(\frac{b_k}{2}\right)^{\lambda_k}}{2^{N-1+\mu} \Gamma(\mu+1)} \sum_{i=0}^{N-1} \sum_{\tau(i, N-1)} \left\{ \prod_{k=1}^{N-1} \left(\frac{2}{b_k}\right)^{2\lambda_k i_k} \Gamma(\lambda_k)^{i_k} \Gamma(-\lambda_k)^{1-i_k} \right\} M_{ij}, \quad (10)$$

where

$$M_{ij} = \int_0^\infty s^\delta K_{\lambda_N}(b_N\sqrt{s}) {}_0F_1\left(; 1 + \mu, -\frac{a^2 s}{4}\right) \prod_{k=1}^{N-1} \left[{}_0F_1\left(; 1 - \lambda_k, \frac{b_k^2 s}{4}\right) \right]^{i_k} \left[{}_0F_1\left(; 1 + \lambda_k, \frac{b_k^2 s}{4}\right) \right]^{1-i_k} ds, \quad (11)$$

whereby j stands for the j -th tuple of the set $\tau(i, N-1)$, and $\delta = \nu + \frac{\mu}{2} + \frac{\sum_{k=1}^{N-1} \lambda_k}{2} - \sum_{k=1}^{N-1} \lambda_k i_k$. The integral M_{ij} can be solved by expressing the Bessel K integrand in terms of Meijer's G-functions [12, Eq. 11], namely, using $K_{\lambda_N}(b_N\sqrt{s}) = \frac{1}{2} G_{0,2}^{2,0} \left(\frac{b_N^2 s}{4} \left| \begin{matrix} - \\ \lambda_N, -\lambda_N \end{matrix} \right. \right)$. Then, considering the change of variable $z = \frac{b_N^2 s}{4}$ yields

$$M_{ij} = \frac{\left(\frac{4}{b_N^2}\right)^{\delta+1}}{2} \int_0^\infty \frac{1}{z} G_{0,2}^{2,0} \left(z \left| \begin{matrix} - \\ \xi, \xi - \lambda_N \end{matrix} \right. \right) {}_0F_1\left(; 1 + \mu, -\frac{a^2 z}{b_N^2}\right) \prod_{k=1}^{N-1} \left[{}_0F_1\left(; 1 - \lambda_k, \frac{b_k^2 z}{b_N^2}\right) \right]^{i_k} \left[{}_0F_1\left(; 1 + \lambda_k, \frac{b_k^2 z}{b_N^2}\right) \right]^{1-i_k} dz, \quad (12)$$

where $\xi = \delta + \frac{\lambda_N}{2} + 1$. By further noticing that a single integral representation for the multivariate Lauricella hypergeometric function $F_C^{(n)}(a, b; c_1, \dots, c_n; x_1, \dots, x_n)$ is given by [9]

$$F_C^{(n)}(a, b; c_1, \dots, c_n; x_1, \dots, x_n) = \frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty G_{0,2}^{2,0}(t|a, b) \left(\prod_{k=1}^n {}_0F_1(; c_k, x_k t) \right) \frac{1}{t} dt, \quad (13)$$

one can easily recognize that $I(\nu, \mu, a, \Lambda, \beta)$ can be expressed in terms of (13) as given in (14) at the top of the next page. Using the multiples series representation of the Lauricella's hypergeometric function (36) in the appendix, it can be noted that the convergence of the total sum involved in (14) is governed by the threshold condition

$$|a^2| < \left(\sqrt{|b_1^2|} + \sqrt{|b_2^2|} + \dots + \sqrt{|b_N^2|} \right)^2. \quad (15)$$

Hopefully, (14) can be extended to the complementary region of (15) by applying the analytical continuation formula of the Lauricella's hypergeometric function F_C (37) shown in the Appendix. Although the derived analytic continuation of (14) is not shown here for lack of space, the former ensures that for all $a > 0$ and all admissible values of $\{b_i\}_{i=1}^N$, the absolute convergence of the series (14) is always guaranteed.

We now simplify (14) in an important case corresponding to $N = 2$. Indeed, for $N = 2$, (14) reduces to

$$I_s(\nu, \mu, a, \lambda_1, \lambda_2, b_1, b_2) = \frac{\left(\frac{a}{b_2}\right)^\mu \left(\frac{b_1}{b_2}\right)^{\lambda_1} \left(\frac{4}{b_2^2}\right)^{\nu+1}}{4\Gamma(\mu+1)} \left\{ \Gamma(\xi_0)\Gamma(\xi_0 - \lambda_2) \Gamma(-\lambda_1) F_4(\xi_0, \xi_0 - \lambda_2, 1 + \mu, 1 + \lambda_1, -\frac{a^2}{b_2^2}, \frac{b_1^2}{b_2^2}) + \left(\frac{b_2}{b_1}\right)^{2\lambda_1} \Gamma(\lambda_1) \Gamma(\xi_1)\Gamma(\xi_1 - \lambda_2) F_4(\xi_1, \xi_1 - \lambda_2, 1 + \mu, 1 - \lambda_1, -\frac{a^2}{b_2^2}, \frac{b_1^2}{b_2^2}) \right\}, \quad (16)$$

$$I(\nu, \mu, a, \Lambda, \beta) = \frac{(\frac{a}{b_N})^\mu \prod_{k=1}^{N-1} (\frac{b_k}{b_N})^{\lambda_k}}{2^N \Gamma(\mu+1)} \left(\frac{4}{b_N^2}\right)^{\nu+1} \sum_{i=0}^{N-1} \sum_{\tau(i, N-1)} \Gamma(\xi) \Gamma(\xi - \lambda_N) \left\{ \prod_{k=1}^{N-1} \left(\frac{b_N}{b_k}\right)^{2\lambda_k} \Gamma(\lambda_k)^{i_k} \Gamma(-\lambda_k)^{1-i_k} \right\} \\ F_C^{(N)}(\xi, \xi - \lambda_N, 1 + \mu, \underbrace{1 - \lambda_1}_{i_1}, \underbrace{1 + \lambda_1}_{1-i_1}, \dots, \underbrace{1 - \lambda_{N-1}}_{i_{N-1}}, \underbrace{1 + \lambda_{N-1}}_{1-i_{N-1}}; -\frac{a^2}{b_N^2}, \frac{b_1^2}{b_N^2}, \dots, \frac{b_{N-1}^2}{b_N^2}). \quad (14)$$

where $\xi_0 = \nu + \frac{\mu}{2} + \frac{\lambda_1 + \lambda_2}{2} + 1$, $\xi_1 = \nu + \frac{\mu}{2} + \frac{\lambda_2 - \lambda_1}{2} + 1$ and $F_4 = F_C^{(2)}$ is the fourth Appell hypergeometric function which is defined as

$$F_4[\alpha, \beta; \gamma, \gamma', x, y] = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\alpha)_{j+k} (\beta)_{j+k} x^j y^k}{(\gamma)_j (\gamma')_k j! k!}, \quad |x|^{1/2} + |y|^{1/2} < 1. \quad (17)$$

The Appell functions [9] are well known, and numerical routines for their exact computation are available in packages such as Mathematica. A special case of I_s corresponds to $b_1 = b_2 = b$ and $\lambda_1 \neq \lambda_2$. In this case, making use of [10, Eq. 7.821.1] along with the identity

$$K_{\lambda_1}(b\sqrt{s}) K_{\lambda_2}(b\sqrt{s}) = \frac{\sqrt{\pi}}{2} G_{2,4}^{4,0} \left(b^2 s \left| \begin{matrix} 0, \frac{1}{2} \\ \frac{\lambda_1 + \lambda_2}{2}, \frac{\lambda_1 - \lambda_2}{2}, \frac{\lambda_2 - \lambda_1}{2}, -\frac{\lambda_1 + \lambda_2}{2} \end{matrix} \right. \right), \quad (18)$$

a closed form of I_s is shown to be given by

$$I_s(\nu, \mu, a, \lambda_1, \lambda_2, b, b) = \frac{\sqrt{\pi}}{2} \left(\frac{4}{a^2}\right)^{\nu+1} G_{4,4}^{4,1} \left(\frac{4b^2}{a^2} \left| \begin{matrix} -\nu - \frac{\mu}{2}, 0, \frac{1}{2}, -\nu + \frac{\mu}{2} \\ \frac{\lambda_1 + \lambda_2}{2}, \frac{\lambda_1 - \lambda_2}{2}, \frac{\lambda_2 - \lambda_1}{2}, -\frac{\lambda_1 + \lambda_2}{2} \end{matrix} \right. \right). \quad (19)$$

III. APPLICATION: ERROR PROBABILITIES FOR AMPLIFY AND FORWARD MULTI-HOP RELAYING SYSTEMS

Consider an N -hop wireless relaying system where the source communicates with the destination node via $N - 1$ amplify and forward variable-gain relays [1]. In such a system, the reciprocal of the end-to-end instantaneous received SNR γ , is the sum of the inverse of the individual per-hop SNRs [5]. We have

$$\gamma = \left[\sum_{l=1}^N \frac{1}{\gamma_l} \right]^{-1}. \quad (20)$$

Then, the moment generating function (MGF) of $\gamma^V = \frac{1}{\gamma}$ is the product of the individual MGFs pertaining to the different hops, thus implying

$$M_{\gamma^V}(s) = \prod_{l=1}^N M_{\frac{1}{\gamma_l}}(s), \quad (21)$$

where $M_{\frac{1}{\gamma_l}}(s)$ is the MGF of the SNR on the l -th hop and can be obtained in closed-form for a variety of fading channel models [7, Eqs. 6-12 and 16]. In Nakagami- m fading, $M_{\gamma^V}(s)$ is given by

$$M_{\gamma^V}(s) = 2^N \left(\prod_{l=1}^N \frac{(m_l)^{m_l/2}}{\Gamma(m_l)} \right) s^{\frac{m_\Sigma}{2}} \prod_{l=1}^N K_{m_l} \left(2\sqrt{\frac{sm_l}{\gamma_l}} \right), \quad (22)$$

whereby $\bar{\gamma}_i = E(\gamma_i)$, with $E(\cdot)$ denoting expectation, $m_l \geq 1/2$ denotes the Nakagami- m factor of the i -th hop and $m_\Sigma = \sum_{l=1}^N m_l$ is defined for the sake of notational convenience.

A. Binary modulations

For different binary modulation schemes, the bit error probability can be expressed in terms of (22) as [8, Eq. 4b]

$$P_e = \frac{1}{2} - \frac{\tau^{\eta/2}}{2\Gamma(\eta)} \int_0^\infty s^{\frac{\eta}{2}-1} J_\eta(2\sqrt{\tau s}) M_{\gamma^V}(s) ds, \quad (23)$$

where the parameters τ and η depend on the type of modulation detection scheme given in [13, Tab. 8.1]. By substituting appropriately (22) in (23) and then using (1), P_e can be written as

$$P_e = \frac{1}{2} - \frac{2^{N-1} \tau^{\eta/2}}{\Gamma(\eta)} \left(\prod_{l=1}^N \frac{(m_l)^{m_l/2}}{\Gamma(m_l)} \right) I \left(\frac{m_\Sigma + \eta}{2} - 1, \eta, 2\sqrt{\tau}, \Lambda, \beta \right), \quad (24)$$

where

$$\Lambda = \{m_1, m_2, \dots, m_N\}, \quad \beta = \{2\sqrt{\frac{m_1}{\gamma_1}}, \dots, 2\sqrt{\frac{m_N}{\gamma_N}}\}, \quad (25)$$

and $I(\cdot, \cdot, \cdot, \cdot, \cdot)$ is defined in (1) and can be obtained by using (14). Equation (24) is a new closed-form expression for the bit error probability of binary modulations in AF relaying systems with variable gain relays under non-identical Nakagami- m fading. In some practical applications, a dual-hop transmission, i.e., $N = 2$, may be sufficient. In this case P_e is expressed in terms of I_s in (16) as

$$P_e = \frac{1}{2} - \frac{(\frac{\tau\bar{\gamma}_2}{m_2})^\eta (\frac{m_1\bar{\gamma}_2^2}{m_2\bar{\gamma}_1})^{m_1}}{\eta B(m_2, \eta)} \left\{ \frac{B(m_1 + \eta, -m_1)}{B(m_2 + \eta, m_1)} F_4(m_1 + m_2 + \eta, m_1 + \eta, 1 + \eta, 1 + m_1, -\frac{\tau\bar{\gamma}_2}{m_2}, \frac{m_1\bar{\gamma}_2^2}{m_2\bar{\gamma}_1}) + (\frac{m_2\bar{\gamma}_1}{m_1\bar{\gamma}_2})^{2m_1} F_4(m_2 + \eta, \eta, 1 + \eta, 1 - m_1, -\frac{\tau\bar{\gamma}_2}{m_2}, \frac{m_1\bar{\gamma}_2^2}{m_2\bar{\gamma}_1}) \right\}, \quad (26)$$

where $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function [10]. It is worthwhile to note that (26) is the first closed-form expression for error probability of binary modulations in two hops AF transmissions operating over non identically distributed Nakagami- m fading channels. So far, previous works have only considered identically distributed Nakagami fading [2], [3], i.e., $m_1 = m_2$ and $\bar{\gamma}_1 = \bar{\gamma}_2$. Another special case, also not handled before, corresponds to identical ratios $\{\frac{m_l}{\gamma_l}\}_{l=1}^2$ across the different hops, a case which includes as well the identically distributed fading as a special case. Defining $\frac{m}{\bar{\gamma}} = \{\frac{m_l}{\gamma_l}\}_{l=1}^2$ and applying (19) yield

$$P_e = \frac{1}{2} - \frac{(\frac{m}{\bar{\gamma}})^{\frac{m_1+m_2}{2}} \sqrt{\pi}}{\Gamma(\eta)\Gamma(m_1)\Gamma(m_2)} G_{4,4}^{4,1} \left(\frac{4m}{\tau\bar{\gamma}} \left| \begin{matrix} 1 - \eta - \frac{m_1+m_2}{2}, 0, \frac{1}{2}, 1 - \frac{m_1+m_2}{2} \\ \frac{m_1+m_2}{2}, \frac{m_1-m_2}{2}, \frac{m_2-m_1}{2}, -\frac{m_1+m_2}{2} \end{matrix} \right. \right). \quad (27)$$

By setting $m = m_1 = m_2$ and $\eta = \tau = 1$, we obtain an equivalent alternative representation for Hasna and Alouini's

main result [2, Eq. 12]. To prove the concordance of the two formulas, we use the Meijer's-G function property in [10, Eq. 9.31.1] along with the identity

$$G_{3,3}^{3,1} \left(z \left| \begin{matrix} a, c, a+1 \\ b, d, a \end{matrix} \right. \right) = \frac{\Gamma(b-1)\Gamma(d-1)}{\Gamma(c-a)} z^a (1 - {}_2F_1(b-a, d-a, c-a, -\frac{1}{z})). \quad (28)$$

For completeness, it is worthwhile to mention that [2, Eq. 12] can also be deduced from (26) by applying the analytical continuation formula of the Lauricella function followed by some algebraic manipulations using the Burchnell formulas [15, Eq. 37],

$$F_4(\alpha, \beta; \gamma, \gamma'; x, x) = {}_4F_3 \left(\alpha, \beta, \frac{1}{2}(\gamma + \gamma'), \frac{1}{2}(\gamma + \gamma' - 1); \gamma, \gamma, \gamma + \gamma' - 1; 4x \right), \quad (29)$$

where ${}_pF_q(\cdot)$ is the generalized hypergeometric function [10, Eq. 9.14.1]. In turn, the generalized hypergeometric function ${}_4F_3$ reduces to a simpler one when its parameters are constrained properly as

$${}_4F_3(a_1, a_2, a_3, a_4; b_1, a_3, a_4, z) = {}_2F_1(a_1, a_2, b_1, z), \quad (30)$$

where $F(a, b, c, z)$ is the Gauss hypergeometric function [10, Eq. 9.14.2]. Hence, applying (37), (29) and (30) to (26), when $m = m_1 = m_2$, $\bar{\gamma} = \bar{\gamma}_1 = \bar{\gamma}_2$ and $\eta = \tau = 1$, yields [2, Eq. 12].

B. M -ary Modulations

The evaluation of the symbol or bit error probabilities induced by coherent M -ary modulation schemes basically involves taking the expectation of the Gaussian- Q function [13, Eq. 4.1] with respect to the SNR pdf. This is the case, for instance, for the evaluation of the bit [14, Eqs. 9 and 10] and symbol [13, Eq. 8.3] error probabilities of M -ary amplitude-shift keying (M -ASK) with Gray encoding and the bit and symbol error probabilities of M -ary quadrature amplitude modulation (M -QAM) with Gray encoding [14, Eqs. 14 and 16] and [13, Eq. 8.9]. In this paper we study the performance of an arbitrary Gray bit-mapped $I \times J$ rectangular QAM constellation. For this type of modulation, the bit error probability is traditionally obtained as

$$P_e = \frac{2}{\log_2(I \cdot J)} \left(\frac{1}{I} \sum_{k=1}^{\log_2(I)} \sum_{i=0}^{(1-2^{-k})I-1} P(i, k) E(Q(\sqrt{2w_i}\gamma)) \right) + \frac{1}{J} \sum_{p=1}^{\log_2(J)} \sum_{j=0}^{(1-2^{-p})J-1} P(j, p) E(Q(\sqrt{2w_j}\gamma)), \quad (31)$$

where $w_k = (2k+1)^2 \frac{3 \log_2(I \cdot J)}{I^2 + J^2 - 2}$, and $P(i, j)$ are expressions of I and J given in [14]. Based on the MGF of the reciprocal of the instantaneous received SNR, the authors of [8] have also provided a single integral form to evaluate the expectation of the Gaussian- Q function given by

$$E(Q(\tau\sqrt{\gamma})) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\sin(\tau\sqrt{2s})}{s} M_{\gamma^V}(s) ds. \quad (32)$$

Inserting (22) into (32) and making use of the identity

$$\sin(\tau\sqrt{2s}) = \sqrt{\frac{\pi\tau s}{2}} J_{\frac{1}{2}}(\tau\sqrt{2s}), \quad (33)$$

the expectation in (32) can be evaluated in closed-form using (1) according to

$$E(Q(b\sqrt{\gamma})) = \frac{1}{2} - \frac{2^{N-1}\sqrt{\tau}}{\sqrt{\pi}\sqrt{2}} \left(\prod_{l=1}^N \frac{(\frac{m_l}{\bar{\gamma}})^{m_l/2}}{\Gamma(m_l)} \right) I\left(\frac{m_\Sigma-3/2}{2}, \frac{1}{2}, \tau\sqrt{2}, \Lambda, \beta\right), \quad (34)$$

where $I(\cdot, \cdot, \cdot, \cdot, \cdot)$ can be obtained by using (14) and Λ, β are given in (25). By substituting (34) into (31) and performing some algebraic manipulations, it follows that

$$P_e = 1 - \frac{2^N}{\sqrt{\pi} \log_2(I \cdot J)} \left(\prod_{l=1}^N \frac{(\frac{m_l}{\bar{\gamma}})^{m_l/2}}{\Gamma(m_l)} \right) \left(\frac{1}{I} \sum_{k=1}^{\log_2(I)} \sum_{i=0}^{(1-2^{-k})I-1} P(i, k) \sqrt{w_i} I\left(\frac{m_\Sigma-3/2}{2}, \frac{1}{2}, 2w_i, \Lambda, \beta\right) + \frac{1}{J} \sum_{p=1}^{\log_2(J)} \sum_{j=0}^{(1-2^{-p})J-1} P(j, p) \sqrt{w_j} I\left(\frac{m_\Sigma-3/2}{2}, \frac{1}{2}, 2w_j, \Lambda, \beta\right) \right). \quad (35)$$

The expression (35) provides a new solution for the evaluation of average error probabilities of modulation schemes involving a Gaussian- Q function in AF multi-hop transmissions with variable gain relays operating over non identical Nakagami- m fading. Using (16) and (19), the error probabilities of M -QAM dual-hop transmissions can be expressed in terms of Appell F_4 or Meijer's-G functions, but for lack of space, are not shown here.

IV. NUMERICAL RESULTS

In this section, we present illustrative numerical examples to validate the analytical expressions obtained in Section III for the evaluation of the error probabilities of AF multihop systems with variable-gain relays. Some exhaustive simulations have been carried out to check the correctness of the analytical formulas. Fig. 1 depicts the bit error probability, given in (24), for dual-hop BPSK ($\eta = \tau = 1$) transmissions over non identical Nakagami-fading. Here the power imbalance between the two hops is investigated. The higher average SNR resulting from one of the relays may be due to a Line Of Sight (LOS) signal component between the source terminal and the relay, or equivalently between the relay and the destination. Such a situation may occur in a cell-site scheme. Note that such imbalance may be either beneficial or harmful for the overall system performance. Indeed, for $\bar{\gamma}_2 > \bar{\gamma}_1$, it is advantageous compared to the balanced case, otherwise, it is detrimental. The influence of the Nakagami- m fading parameters on the performance is also analyzed. Fig. 2 sketches the bit error probabilities for different QPSK multihop transmission systems, obtained from (35), for different numbers of relays. We assume without loss of generality that the channels are independent and identically distributed, i.e., $m_i = m$ and $\bar{\gamma}_i = \bar{\gamma}$. As expected, a substantial improvement in performance can be attained from the increase of the Nakagami fading parameter m . On the other hand, the number of relays has a less pronounced effect on the error performance.

V. CONCLUSION

In this paper, new closed-form expressions for the error probabilities of binary and M -ary modulations are derived when such schemes are used along with AF multi-hop systems

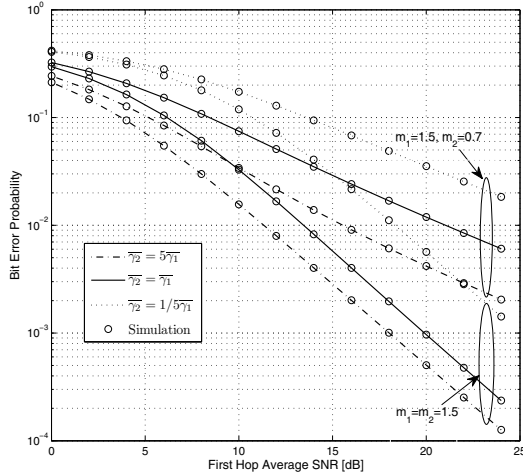


Fig. 1. Bit error probabilities for different BPSK AF multihop transmission systems with variable-gain relays over non necessarily identical Nakagami fading.

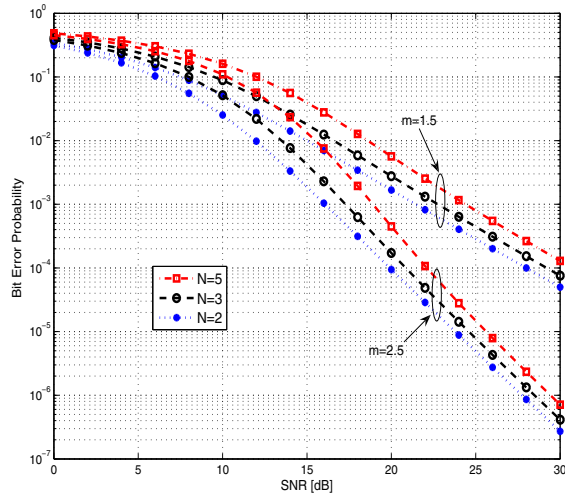


Fig. 2. Average SEP vs. average SNR for QPSK and different values of the fading parameter and number of hops $N = 2, 3, 5$.

with variable gain relays operating over independent but non-necessarily identically distributed Nakagami fading channels. The obtained formulas establish a connection, hitherto unknown, between the error probabilities and the Lauricella's functions. Our analysis results extend previously known results in the literature for special cases of interest. Although this work focuses on the performance of multihop AF relaying in terms of error probability, this analysis may be extended to cover other useful performance measures, namely, the ergodic capacity.

APPENDIX

The third Lauricella function $F_C^{(n)}(a, b; c_1, \dots, c_n; x_1, \dots, x_n)$ of n variables is defined

as the multiple hypergeometric series [9, Eq. (A.1.4)]

$$F_C^{(n)}(a, b; c_1, \dots, c_n; x_1, \dots, x_n) = \sum_{m_1=0}^{\infty} \dots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n} (b)_{m_1+\dots+m_n} x_1^{m_1} \dots x_n^{m_n}}{(c_1)_{m_1} \dots (c_n)_{m_n} m_1! \dots m_n!}; \sqrt{|x_1|} + \dots + \sqrt{|x_n|} < 1, \quad (36)$$

where $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$ denotes the Pochhammer symbol.

A Lauricella function in the argument z_i can be analytically continued to a sum of two Lauricella functions in the argument $x_i = z_i/z_n$ for $i = 1, \dots, n-1$ and $x_n = 1/z_n$ by the following relation [9]

$$F_C^{(n)}(a, b; c_1, \dots, c_n; z_1, \dots, z_n) = \frac{\Gamma(c_n)\Gamma(b-a)}{\Gamma(b)\Gamma(c_n-a)} (-z_n)^{-a} F_C^{(n)}(a, 1+a-c_n; c_1, \dots, c_{n-1}, 1-b+a; x_1, \dots, x_n) + \frac{\Gamma(c_n)\Gamma(a-b)}{\Gamma(a)\Gamma(c_n-b)} (-z_n)^{-b} F_C^{(n)}(b, 1+b-c_n; c_1, \dots, c_{n-1}, 1-a+b; x_1, \dots, x_n). \quad (37)$$

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