Closed-Form Expression for the Exact Cramér-Rao Bound of Timing Recovery Estimators from MSK Transmissions

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***Abstract*—This paper addresses the problem of time delay estimation of minimum shift keying (MSK)-modulated signals. We derive for the first time the analytical expression of the stochastic Cramér-Rao lower bound (CRLB) for non-data-aided (NDA) timing recovery, from which the effects of the shaping pulse are easily observed. It is assumed that the transmitted symbols are completely unknown at the receiver. Moreover, the carrier phase and frequency are considered as unknown nuisance parameters. We emphasize the penalty resulting from considering the transmitted data as unknown in the estimation of the time delay.**

***Index Terms*—MSK signals, symbol timing recovery, Cramér-Rao lower bound (CRLB), non data aided (NDA) estimation.**

I. Introduction

In modern communication systems, parameter estimation is a crucial task for any digital receiver; in particular the recovery of the time delay introduced by the channel, which arises in various applications such as radar, sonar or wireless communications. In this context, many estimators have been developed to accurately recover the time delay for MSK-modulated signals [1, 2]. Actually, NDA schemes have been extensively studied since they do not impinge on the whole throughput of the system. It is then of interest to derive a benchmark for the performance of such estimators. The well known CRLB meets this requirement and is usually used to evaluate the performance of actual unbiased estimators [3, 4]. Nevertheless, it is often difficult if not impossible to derive a closed-form expression for the stochastic CRLB which it is usually empirically computed. Yet, many works have dealt with the derivation of this lower bound but under different simplifying assumptions. Most of them consider the transmitted data to be perfectly known to the receiver (data-aided CRLB) or derive the modified CRLB (MCRLB) [5, 6] which is actually a looser bound especially in the low signal-to-noise ratio (SNR) region. The MCRLB assumes the transmitted symbols to be unknown but it cannot be achieved by practical estimators unlike the stochastic CRLB. Other works tackled this problem but for special SNR regions (very-low/high SNRs) and hence the derived bound stands valid only asymptotically (ACRLB) [8].

To the best of our knowledge, no contribution has dealt so far with stochastic CRLBs for time delay estimates of MSK-modulated signals, although this type of modulation is widely used especially in GSM systems[1]. Motivated by these facts, in this work, we derive for the first time the analytical expression of the CRLB for NDA signal timing recovery from MSK-modulated signals. We treat the general case in which the carrier phase and frequency offset are completely unknown at the receiver, and we show that the problem of time delay estimation and the carrier phase and frequency offset estimations are two decoupled problems (the global Fisher Information matrix is block diagonal).

This paper is organized as follows. In Section II, we introduce the system model. In Section III, we prove the block diagonal structure of the Fisher information matrix. In Section IV, we derive the stochastic CRLB for time delay estimates of MSK-modulated signals. Some graphical representations are presented in Section V and, finally, some concluding remarks are drawn out in Section VI.

II. System Model

Consider a traditional communication system where the channel delays the transmitted signal and an additive white Gaussian noise (AWGN) corrupts the received signal. In the case of imperfect phase and frequency synchronizations, the received signal can be expressed as:

$$y(t) = \sqrt{E_s}\ x(t-\tau)e^{j(2\pi f_c t+\theta)} + w(t), \qquad (1)$$

where τ is the time delay to be estimated, θ is the channel distortion phase and f_c is the carrier frequency offset. The parameters τ, θ and f_c are assumed to be deterministic but unknown and we gather all these unknown parameters in one vector:

$$\boldsymbol{\nu} = [\tau, \theta, f_c]^T. \qquad (2)$$

In (1), $w(t)$ is an AWGN with independent real and imaginary parts, each of variance σ^2, and $x(t)$ stands for the transmitted signal given by:

$$x(t) = \sum_{i=1}^{K} a_i\ h(t - i\ T), \qquad (3)$$

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[1] GSM transmissions are based on the Gaussian MSK modulation, where the pulse train is filtered prior to MSK modulation.

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with $\{a_i\}_{i=1}^{K}$ being the sequence of K unknown transmitted symbols drawn from an MSK constellation[2] and T is the symbol duration. The transmitted symbols are assumed to be statistically independent and equally likely, with normalized energy, i.e., $E\{|a_i|^2\} = 1$. Finally, $h(t)$ is the symbol-shaping function which will be seen to have an important impact on the CRLB and therefore on the system's performance.

Actually the CRLB on the joint estimation of τ, θ and f_c (i.e., $\boldsymbol{\nu}$) is defined as [3, 4]:

$$\text{CRLB}(\boldsymbol{\nu}) = \boldsymbol{I}^{-1}(\boldsymbol{\nu}), \tag{4}$$

where the entries of the Fisher information matrix (FIM), $\boldsymbol{I}(\boldsymbol{\nu})$, are defined as:

$$[\boldsymbol{I}(\boldsymbol{\nu})]_{i;j} = E\left\{\frac{\partial L(\boldsymbol{\nu})}{\partial \nu_i}\,\frac{\partial L(\boldsymbol{\nu})}{\partial \nu_j}\right\}\ ,\ i,\ j = 1,2,3, \tag{5}$$

with $L(\boldsymbol{\nu})$ being the log-likelihood function of the parameters to be estimated, $\{\nu_i\}_{i=1}^{3}$ are the elements of $\boldsymbol{\nu}$ and $E\{.\}$ stands for the statistical expectation.

The following notations are used in the rest of this paper: $|.|$, $\Re\{.\}$, $\Im\{.\}$ and $\{.\}^*$ stand for complex magnitude, real, imaginary and conjugate, respectively and j is the complex number defined as $j^2 = -1$. Moreover, we define the SNR of the system as: $\rho = E_s/2\sigma^2$.

III. Block Diagonal Structure of the FIM

In order to derive the time delay CRLB, CRLB(τ), we need to inverse the global FIM matrix, and this admittedly requires the knowledge of all its entries. However, when the parameter of interest is decoupled from the other nuisance parameters, the derivation of its CRLB reduces simply to the computation of its corresponding diagonal element in the FIM. Fortunately, as we will prove subsequently in this section, the problem of estimating the time delay is indeed decoupled from that of estimating the carrier phase and frequency offset.

To begin with, under the assumptions made so far, it can be seen that the likelihood function of the received signal can be written as [5, 9]:

$$\begin{aligned}\Lambda(y(t);\boldsymbol{\nu}) \;=\; & E\left\{\exp\left\{\frac{\sqrt{E_s}}{\sigma^2}\int_{-\infty}^{+\infty}\Re\left\{\tilde{y}(t)x^*(t)\right\}dt - \right.\right.\\ & \left.\left.\frac{E_s}{2\sigma^2}\int_{-\infty}^{+\infty}|x(t)|^2 dt\right\}\right\},\end{aligned} \tag{6}$$

where the expectation is performed with respect to the transmitted symbols and $\tilde{y}(t)$ is the virtually derotated received signal given by:

$$\widetilde{y}(t) = y(t)\ e^{-j(2\pi f_c t+\theta)}. \tag{7}$$

Note that since the transmitted symbols $\{a_i\}_{i=1}^{K}$ are equally likely, then the likelihood function, $\Lambda(y(t);\boldsymbol{\nu})$, is given by:

$$\Lambda(y(t);\boldsymbol{\nu}) = \frac{1}{M^K}\prod_{i=1}^{K} H_i(\boldsymbol{\nu}), \tag{8}$$

[2]In MSK transmissions, the symbols are defined as $a_{k+1} = j a_k c_k$ where c_k is a sequence of BPSK symbols and a_0 is the original value drawn from the set $\{-1,\ -j,\ +1,\ +j\}$ [7].

with:

$$\begin{aligned}H_i(\boldsymbol{\nu}) \;=\; & \sum_{b_k\in C}\exp\left\{-\frac{E_s}{2\sigma^2}|b_k|^2\right\}\times\\ & \exp\left\{\frac{\sqrt{E_s}}{\sigma^2}\int_{-\infty}^{+\infty}\Re\left\{\widetilde{y}(t)b_k^*\right\}h(t-iT-\tau)dt\right\},\end{aligned} \tag{9}$$

where C is the constellation alphabet. Using (8), we write the log-likelihood function, $L(\boldsymbol{\nu}) = \ln(\Lambda(y(t);\boldsymbol{\nu}))$, of the received signal as follows:

$$L(\boldsymbol{\nu}) = -K\ln(M) + \sum_{i=1}^{K}\ln(H_i(\boldsymbol{\nu})). \tag{10}$$

Actually, due to the complexity of the log-likelihood function, the analytical derivation of the time delay stochastic CRLB for any constellation is much easier said than done. However, considering the special case of MSK-modulated signals, we are able to manipulate the log-likelihood function in a way that allows the derivation of the analytical expression for CRLB(τ).

First, noting that $|b_i| = 1$ for every $b_i \in C$ and dropping the terms that do not depend on $\boldsymbol{\nu}$, the log-likelihood function of interest, $L(\boldsymbol{\nu})$, reduces simply to:

$$\begin{aligned}L(\boldsymbol{\nu}) \;=\; & \sum_{i=1}^{K}\ln\left(\cosh\left(\frac{\sqrt{E_s}}{\sigma^2}\int_{-\infty}^{+\infty}\Re\{(j^{i-1}a_0)^*\widetilde{y}(t)\}\right.\right.\\ & \left.\left. h(t-iT-\tau)dt\right)\right),\end{aligned} \tag{11}$$

with $\cosh(a) = (e^a + e^{-a})/2$. In the sequel, we assume, without loss of generality, that $a_0 = 1$ and K is an even number, i.e., $K = 2P$. These assumptions are considered just to make the derivations more manageable[3]. Then the log-likelihood function can be rewritten as follows:

$$L(\boldsymbol{\nu}) = \sum_{i=1}^{P}\ln\left(\cosh\left(\frac{\sqrt{E_s}}{\sigma^2}U_i(\boldsymbol{\nu})\right)\right) + \ln\left(\cosh\left(\frac{\sqrt{E_s}}{\sigma^2}V_i(\boldsymbol{\nu})\right)\right), \tag{12}$$

where

$$U_i(\boldsymbol{\nu}) = \int_{-\infty}^{+\infty}\Re\{\tilde{y}(t)\}\ h(t-(2i-1)T-\tau)dt, \tag{13}$$

and

$$V_i(\boldsymbol{\nu}) = \int_{-\infty}^{+\infty}\Im\{\tilde{y}(t)\}\ h(t-2iT-\tau)dt. \tag{14}$$

Starting from (12), we show in Appendix A that the estimation of the time delay is disjoint from the estimation of the carrier phase and frequency offset. In other words, we show that the FIM is block-diagonal structured as follows:

$$\boldsymbol{I}(\boldsymbol{\nu}) = \begin{pmatrix}\text{CRLB}^{-1}(\tau) & \mathbf{0}\\ \mathbf{0} & \boldsymbol{I}_2(\theta, f_c)\end{pmatrix}, \tag{15}$$

[3]Note that the same derivation steps stand valid when $a_0 \neq 1$.

where $\mathbf{0} = [0\ ,\ 0]^T$, CRLB(τ) is the CRLB of the time delay parameter and $\boldsymbol{I}_2(\theta, f_c)$ is the (2×2) FIM pertaining to the joint estimation of f_c and θ.

IV. Time Delay CRLBs for MSK-Modulated Signals

Now, since the estimation problems are decoupled, we only need to derive the first diagonal element of the FIM in order to find CRLB(τ) under imperfect frequency and phase synchronizations. In fact, the first derivative of (12) with respect to τ is:

$$\frac{\partial L(\boldsymbol{\nu})}{\partial \tau} = \frac{\sqrt{E_s}}{\sigma^2}\left[\sum_{i=1}^{P} \frac{\sinh\left(\frac{\sqrt{E_s}}{\sigma^2}U_i(\boldsymbol{\nu})\right)}{\cosh\left(\frac{\sqrt{E_s}}{\sigma^2}U_i(\boldsymbol{\nu})\right)}\dot{U}_i(\boldsymbol{\nu}) + \sum_{i=1}^{P} \frac{\sinh\left(\frac{\sqrt{E_s}}{\sigma^2}V_i(\boldsymbol{\nu})\right)}{\cosh\left(\frac{\sqrt{E_s}}{\sigma^2}V_i(\boldsymbol{\nu})\right)}\dot{V}_i(\boldsymbol{\nu})\right], \quad (16)$$

with $\dot{U}_i(\boldsymbol{\nu})$ and $\dot{V}_i(\boldsymbol{\nu})$ being the derivatives of $U_i(\boldsymbol{\nu})$ and $V_i(\boldsymbol{\nu})$ with respect to τ, respectively. Then, the first diagonal element of the FIM matrix is expressed as:

$$\begin{aligned}[\boldsymbol{I}(\boldsymbol{\nu})]_{1;1} &= \mathrm{E}\left\{\left(\frac{\partial L(\boldsymbol{\nu})}{\partial \tau}\right)^2\right\} \\ &= \frac{E_s}{\sigma^4}\,\mathrm{E}\left\{\sum_{i=1}^{P}\sum_{l=1}^{P}\left[\frac{\sinh\left(\frac{\sqrt{E_s}}{\sigma^2}U_i(\boldsymbol{\nu})\right)}{\cosh\left(\frac{\sqrt{E_s}}{\sigma^2}U_i(\boldsymbol{\nu})\right)} \times \frac{\sinh\left(\frac{\sqrt{E_s}}{\sigma^2}U_l(\boldsymbol{\nu})\right)}{\cosh\left(\frac{\sqrt{E_s}}{\sigma^2}U_l(\boldsymbol{\nu})\right)}\,\dot{U}_i(\boldsymbol{\nu})\dot{U}_l(\boldsymbol{\nu}) + 2\frac{\sinh\left(\frac{\sqrt{E_s}}{\sigma^2}U_i(\boldsymbol{\nu})\right)}{\cosh\left(\frac{\sqrt{E_s}}{\sigma^2}U_i(\boldsymbol{\nu})\right)} \times \right.\right. \\ &\left.\left. \frac{\sinh\left(\frac{\sqrt{E_s}}{\sigma^2}V_l(\boldsymbol{\nu})\right)}{\cosh\left(\frac{\sqrt{E_s}}{\sigma^2}V_l(\boldsymbol{\nu})\right)}\,\dot{U}_i(\boldsymbol{\nu})\dot{V}_l(\boldsymbol{\nu}) + \frac{\sinh\left(\frac{\sqrt{E_s}}{\sigma^2}V_i(\boldsymbol{\nu})\right)}{\cosh\left(\frac{\sqrt{E_s}}{\sigma^2}V_i(\boldsymbol{\nu})\right)} \times \frac{\sinh\left(\frac{\sqrt{E_s}}{\sigma^2}V_l(\boldsymbol{\nu})\right)}{\cosh\left(\frac{\sqrt{E_s}}{\sigma^2}V_l(\boldsymbol{\nu})\right)}\,\dot{V}_i(\boldsymbol{\nu})\dot{V}_l(\boldsymbol{\nu})\right]\right\}. \end{aligned} \quad (17)$$

At this stage, we need some useful statistical properties of the different random variables involved in (17). We show in Appendix B that $U_i(\boldsymbol{\nu})$ and $\dot{U}_i(\boldsymbol{\nu})$ are statistically independent, as well as $V_i(\boldsymbol{\nu})$ and $\dot{V}_i(\boldsymbol{\nu})$. And using the fact that $U_i(\boldsymbol{\nu})$ and $V_i(\boldsymbol{\nu})$ are identically distributed (see Appendix B), $[\boldsymbol{I}(\boldsymbol{\nu})]_{1;1}$ reduces simply to:

$$[\boldsymbol{I}(\boldsymbol{\nu})]_{1;1} = \frac{2E_s}{\sigma^4}\sum_{i=1}^{P}\sum_{l=1}^{P}\tanh\left(\frac{\sqrt{E_s}}{\sigma^2}U_i(\boldsymbol{\nu})\right) \times \tanh\left(\frac{\sqrt{E_s}}{\sigma^2}U_l(\boldsymbol{\nu})\right)\dot{U}_i(\boldsymbol{\nu})\dot{U}_l(\boldsymbol{\nu}), \quad (18)$$

with $\tanh(.)$ being the hyperbolic tangent function defined as $\tanh(a) = \sinh(a)/\cosh(a)$. In the sequel, we separate the case where $i = l$ and $i \neq l$ in (18). First, consider $i = l$ and since $U_i(\boldsymbol{\nu})$ and $\dot{U}_i(\boldsymbol{\nu})$ are independent, we can write:

$$E\left\{\tanh^2\left(\frac{\sqrt{E_s}}{\sigma^2}U_i(\boldsymbol{\nu})\right)\dot{U}_i^2(\boldsymbol{\nu})\right\} = E\left\{\tanh^2\left(\frac{\sqrt{E_s}}{\sigma^2}U_i(\boldsymbol{\nu})\right)\right\} \times E\left\{\dot{U}_i^2(\boldsymbol{\nu})\right\}. \quad (19)$$

From the definition of $U_i(\boldsymbol{\nu})$ in (13), $\dot{U}_i(\boldsymbol{\nu})$ is given by:

$$\dot{U}_i(\boldsymbol{\nu}) = \sqrt{E_s}\sum_{k=1}^{K}\Re\{a_k\}\dot{g}((i-k)T) - \int_{-\infty}^{+\infty}\Re\{\tilde{w}(t)\}\dot{h}(t - iT - \tau)dt, \quad (20)$$

where $\tilde{w}(t)$ is the derotated noise (i.e., $\tilde{w}(t) = w(t)\exp\{j(2\pi f_c + \theta)\}$) and $g(.)$ is the Nyquist pulse defined as:

$$g(t) = \int_{\infty}^{+\infty} h(u)h(t+u)du, \quad (21)$$

and $\dot{g}(.)$ its first derivative. Now, using (20) and the distribution of $U_i(\boldsymbol{\nu})$ introduced in (39), it can be shown that the expectations involved in (19) reduce to:

$$E\left\{\dot{U}_i^2(\boldsymbol{\nu})\right\} = \sqrt{E_s}2\sum_{k=1}^{K}\dot{g}^2((i-k)T) - \sigma^2\ddot{g}(0), \quad (22)$$

$$\begin{aligned} E\left\{\tanh^2\left(\frac{\sqrt{E_s}}{\sigma^2}U_i(\boldsymbol{\nu})\right)\right\} &= \frac{e^{-\rho}}{\sqrt{2\pi}}\left[\int_{-\infty}^{+\infty}\cosh\left(\sqrt{2\rho}\ x\right) \times e^{-\frac{x^2}{2}}dx - \beta(\rho)\right] \\ &= 1 - \frac{e^{-\rho}}{\sqrt{2\pi}}\beta(\rho). \end{aligned} \quad (23)$$

where $\beta(.)$ is defined as:

$$\beta(\rho) = \int_{-\infty}^{+\infty}\frac{e^{-\frac{x^2}{2}}}{\cosh\left(\sqrt{2\rho}\ x\right)}dx. \quad (24)$$

The treatement of the case $i \neq l$ differs slightly from the treatement done above. In fact, there is a statistical dependence between $U_i(\boldsymbol{\nu})$, $\dot{U}_i(\boldsymbol{\nu})$ and $\dot{U}_l(\boldsymbol{\nu})$ because of the intersymbol interference. Therefore, we first condition the expectation on $U_i(\boldsymbol{\nu})$ and $U_l(\boldsymbol{\nu})$, then we average the result with respect to these two random variables. Indeed, noting that:

$$E\{\dot{U}_i(\boldsymbol{\nu})|U_i(\boldsymbol{\nu}), U_l(\boldsymbol{\nu})\} = U_l(\boldsymbol{\nu})\dot{g}((i-l)T), \quad (25)$$

it follows directly that:

$$\begin{aligned} &E\left\{\tanh\left(\frac{\sqrt{E_s}}{\sigma^2}U_i(\boldsymbol{\nu})\right)\tanh\left(\frac{\sqrt{E_s}}{\sigma^2}U_l(\boldsymbol{\nu})\right) \times \dot{U}_i(\boldsymbol{\nu})\dot{U}_l(\boldsymbol{\nu})|U_i(\boldsymbol{\nu}), U_l(\boldsymbol{\nu})\right\} \\ &= \tanh\left(\frac{\sqrt{E_s}}{\sigma^2}U_i(\boldsymbol{\nu})\right)\tanh\left(\frac{\sqrt{E_s}}{\sigma^2}U_l(\boldsymbol{\nu})\right)U_i(\boldsymbol{\nu})U_l(\boldsymbol{\nu}) \\ &\quad \dot{g}((i-l)T)\dot{g}((l-i)T). \end{aligned} \quad (26)$$

Now considering the fact that $U_i(\boldsymbol{\nu})$ and $U_l(\boldsymbol{\nu})$ are independent, we obtain:

$$\begin{aligned} &E\left\{\tanh\left(\frac{\sqrt{E_s}}{\sigma^2}U_i(\boldsymbol{\nu})\right)\tanh\left(\frac{\sqrt{E_s}}{\sigma^2}U_l(\boldsymbol{\nu})\right)\dot{U}_i(\boldsymbol{\nu})\dot{U}_l(\boldsymbol{\nu})\right\} = \\ &-\left(E\left\{\tanh\left(\frac{\sqrt{E_s}}{\sigma^2}U_i(\boldsymbol{\nu})\right)U_i(\boldsymbol{\nu})\right\}\right)^2\dot{g}^2((i-l)T). \end{aligned} \quad (27)$$

Finally, gathering the results presented in (22), (23) and (27) and after some algebraic manipulations, we obtain CRLB(τ) for MSK-modulated signals, as follows:

$$\text{CRLB}=\frac{1}{4\rho}\left[\left(1-\frac{1}{\sqrt{2\pi}}e^{-\rho}\beta(\rho)\right)\left(\rho\sum_{m=1}^{K}\sum_{n=1}^{K}\dot{g}((m-n)T)-\frac{K}{2}\ddot{g}(0)\right)-\rho\sum_{i=1}^{K}\sum_{l=1}^{K}\dot{g}^2((i-l)T)\right]^{-1}, \quad (28)$$

We note here that the CRLB for time delay estimation is a function of the SNR and the shape of the transmitted pulse is involved through the quantities[4] $\dot{g}((i-l)T)$ and $\ddot{g}(0)$. Besides, our new analytical expression reveals that τ does not appear in CRLB(τ). Hence, the achievable performance holds irrespectively of the true time delay parameter as intuitively expected.

V. Graphical Representations

In this section, we include graphical representations of the lower bound given by (28). First, we verify that the integrant function in (24) takes extremely small values for $|x| > 40$. Therefore the infinite integral in (24) can be accurately approximated by a finite integral over $[-40,\ 40]$ for which the Riemann integration method can be adequately used. In Fig. 1, we plot the CRLB and the MCRLB, for a root-raised cosine pulse with roll-off factors of 0.2 and 1. Clearly, timing estimation is less accurate at a lower roll-off factor where the intersymbol interference is more important. Moreover, we see that the curves representing the CRLB and the MCRLB coincide at high SNR values. Therefore, in this SNR region, the achievable performance in the NDA mode is equivalent to the one obtained when the data are perfectly known since in this SNR region, the MCRLB itself coincides with the DA CRLB. However, in the low-SNR region, the MCRLB departs dramatically from the stochastic CRLB confirming thereby that it is only a loose bound and does not reflect the actual achievable performance.

VI. Conclusion

In this paper, we derived for the first time an analytical expression for the exact CRLB on the variance of NDA time delay estimators from MSK-modulated signals. Moreover, we proved analytically that the time delay estimation is decoupled from the estimation of the phase and the frequecy offset; meaning that the knowledge of these nuisance parameters does not bring any additional information about the unknown time delay to be estimated.

Appendix A
Proof of the Block-Diagonal Structure of the FIM

Recall that we want to prove that τ and $\boldsymbol{u} = [\theta,\ f_c]$ are decoupled. To that end, we use $\frac{\partial L(\boldsymbol{\nu})}{\partial \tau}$ in (16) and we differentiate the log-likelihood function in (12) with respect to the elements $\{u(l)\}_{l=1}^{2}$ of $\boldsymbol{u}$:

$$\frac{\partial L(\boldsymbol{\nu})}{\partial \boldsymbol{u}(l)} = \frac{\sqrt{E_s}}{\sigma^2}\left[\sum_{i=1}^{P}\frac{\sinh\left(\frac{\sqrt{E_s}}{\sigma^2}U_i(\boldsymbol{\nu})\right)}{\cosh\left(\frac{\sqrt{E_s}}{\sigma^2}U_i(\boldsymbol{\nu})\right)}\frac{\partial U_i(\boldsymbol{\nu})}{\partial \boldsymbol{u}(l)}+\sum_{i=1}^{P}\frac{\sinh\left(\frac{\sqrt{E_s}}{\sigma^2}V_i(\boldsymbol{\nu})\right)}{\cosh\left(\frac{\sqrt{E_s}}{\sigma^2}V_i(\boldsymbol{\nu})\right)}\frac{\partial V_i(\boldsymbol{\nu})}{\partial \boldsymbol{u}(l)}\right],\ l=1,\ 2. \quad (29)$$

Then we average $\frac{\partial L(\boldsymbol{\nu})}{\partial \tau}\frac{\partial L(\boldsymbol{\nu})}{\partial \boldsymbol{u}(l)}$, and following the same steps from (17) to (18), we obtain the following result:

$$[\boldsymbol{I}(\boldsymbol{\nu})]_{1;l+1} = \text{E}\left\{\frac{\partial L(\boldsymbol{\nu})}{\partial \tau}\frac{\partial L(\boldsymbol{\nu})}{\partial \boldsymbol{u}(l)}\right\} = 2\frac{E_s}{\sigma^4}\sum_{i=1}^{P}\sum_{m=1}^{P}E\left\{\tanh\left(\frac{\sqrt{E_s}}{\sigma^2}U_i(\boldsymbol{\nu})\right)\times\tanh\left(\frac{\sqrt{E_s}}{\sigma^2}U_m(\boldsymbol{\nu})\right)\frac{\partial U_i(\boldsymbol{\nu})}{\partial \boldsymbol{u}(l)}\frac{\partial U_m(\boldsymbol{\nu})}{\partial \tau}\right\}. \quad (30)$$

To simplify the calculations, we consider $l = 1$ (i.e., we differentiate with respect to f_c):

$$\frac{\partial U_i(\boldsymbol{\nu})}{\partial f_c} = 2\pi\int_{-\infty}^{+\infty}t\Im\left\{y(t)e^{-j(2\pi f_c t+\theta)}\right\}h(t-(2i-1)T-\tau)dt = 2\pi\left[\sqrt{E_s}\sum_{k=1}^{K}\Im\{a_k\}\int_{-\infty}^{+\infty}th(t-(2i-1)T-\tau)\times h(t-kT-\tau)dt+\int_{-\infty}^{+\infty}t\Im\{\tilde{w}(t)\}h(t-(2i-1)T-\tau)dt\right]. \quad (31)$$

Obviously, $\frac{\partial U_i(\boldsymbol{\nu})}{\partial f_c}$ is a function of the imaginary part of the transmitted symbols and the derotated noise only. Thus, it is independent from $U_i(\boldsymbol{\nu})$ and $\frac{\partial U_i(\boldsymbol{\nu})}{\partial \tau}$. As a result, we are able

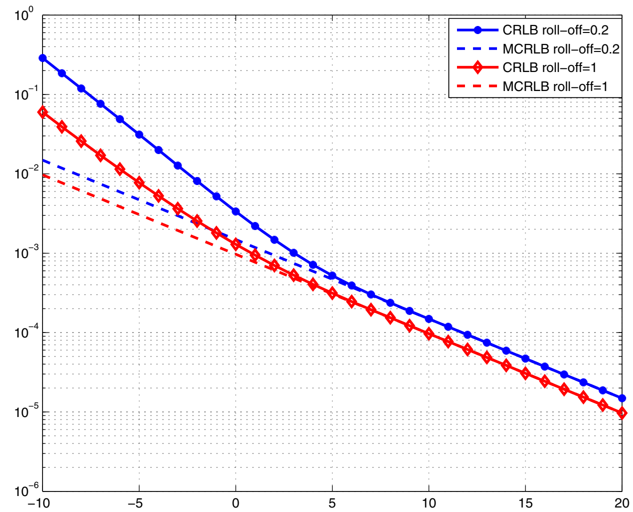

Fig. 1. CRLB(τ) vs. SNR for a root raised cosine pulse shape and different values of the roll-off factor.

[4]For large values of K, we can approximate $\sum_{i=1}^{K}\sum_{l=1}^{K}\dot{g}^2((i-l)T)$ by $K\sum_{i=-\infty}^{+\infty}\dot{g}^2(iT)$ [8].

to split the expectation in (30) into two terms: one involving $U_i(\boldsymbol{\nu})$ and $\frac{\partial U_i(\boldsymbol{\nu})}{\partial \tau}$ and the other involving $\frac{\partial U_i(\boldsymbol{\nu})}{\partial f_c}$. Moreover, we verify that the expectation of $\frac{\partial U_i(\boldsymbol{\nu})}{\partial f_c}$ is equal to zero. Thus, we have:

$$\begin{aligned}[\boldsymbol{I}(\boldsymbol{\nu})]_{1;2} &= \sum_{i=1}^{P}\sum_{m=1}^{P} E\Bigg\{ \tanh\left(U_i(\boldsymbol{\nu})\right)\tanh\left(U_m(\boldsymbol{\nu})\right) \times \\ &\quad \frac{\partial U_i(\boldsymbol{\nu})}{\partial \tau}\Bigg\} E\left\{\frac{\partial U_m(\boldsymbol{\nu})}{\partial f_c}\right\} \\ &= 0. \end{aligned} \tag{32}$$

Hence we proof here that the two parameters τ and f_c are decoupled. The same manipulations are used to prove that τ and θ are also decoupled.

Appendix B

A - Proof of Statistical Independence of $U_i(\boldsymbol{\nu})$ and $\dot{U}_i(\boldsymbol{\nu})$

To begin with, note that $U_i(\boldsymbol{\nu})$ can be written in the following form:

$$U_i(\boldsymbol{\nu}) = \alpha_i + \beta_i(\boldsymbol{\nu}), \tag{33}$$

where

$$\alpha_i = \sqrt{E_s}\Re\{a_i\}, \tag{34}$$

and

$$\beta_i(\boldsymbol{\nu}) = \int_{-\infty}^{+\infty} \Re\{w(t)e^{-j(2\pi f_c t+\theta)}\} h(t-(2i-1)T-\tau)dt. \tag{35}$$

Therefore, $\dot{U}_i(\boldsymbol{\nu})$ is given by:

$$\begin{aligned}\dot{U}_i(\boldsymbol{\nu}) &= \sqrt{E_s}\sum_{m=1}^{K}\Re\{a_m\}\dot{g}((i-m)T) - \\ &\quad \int_{-\infty}^{+\infty}\Re\{w(t)e^{-j(2\pi f_c t+\theta)}\}\dot{h}(t-(2i-1)T-\tau)dt \\ &= \sqrt{E_s}\sum_{m=1}^{K}\Re\{a_m\}\dot{g}((i-m)T) - \dot{\beta}_i(\boldsymbol{\nu}). \end{aligned} \tag{36}$$

In addition, α_i and $\dot{\beta}_i(\boldsymbol{\nu})$ are independent since the noise and the transmitted symbols are independent. Moreover, the term wheighting $\Re\{a_i\}$ in $\sum_{m=1}^{K}\Re\{a_m\}\dot{g}((i-m)T)$ is equal to zero ($\dot{g}(0) = 0$), therefore, $\Re\{a_i\}$ and $\sum_{m=1}^{K}\Re\{a_m\}\dot{g}((i-m)T)$ are also independent. In addition, $\beta_i(\boldsymbol{\nu})$ and $\dot{\beta}_i(\boldsymbol{\nu})$ are Gaussian (obtained from linear transformations of a Gaussian process) and their cross-correlation is equal to zero, as shown below:

$$\begin{aligned}\mathrm{E}\{\beta_i(\boldsymbol{\nu})\dot{\beta}_i(\boldsymbol{\nu})\} &= \mathrm{E}\Bigg\{\iint_{-\infty}^{+\infty}\Re\{\tilde{w}(t_1)\} \times \\ &\quad \Re\{\tilde{w}(t_2)\}h(t_1-iT-\tau)\dot{h}(t_2-iT-\tau)dt_1dt_2\Bigg\} \\ &= \frac{\sigma^2}{2}\iint_{-\infty}^{+\infty}\delta(t_1-t_2)h(t_1)\dot{h}(t_2)dt_1dt_2 \\ &= \frac{\sigma^2}{2}\dot{g}(0) \\ &= 0. \end{aligned} \tag{37}$$

Hence $\beta_i(\boldsymbol{\nu})$ and $\dot{\beta}_i(\boldsymbol{\nu})$ are actually two uncorrelated Gaussian random processes and therefore independent. Thus, $U_i(\boldsymbol{\nu})$ and $\dot{U}_i(\boldsymbol{\nu})$ are independent.

B - PDF of $U_i(\boldsymbol{\nu})$

It is easy to see that the pdf of $U_i(\boldsymbol{\nu})$ conditioned on the transmitted data is:

$$\begin{aligned}P(U_i(\boldsymbol{\nu})|a_{2i-1}) &= \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{\left(U_i(\boldsymbol{\nu})-\sqrt{E_s}\Re\{a_{2i-1}\}\right)^2}{2\sigma^2}\right\} \\ &= \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{\frac{-U_i^2(\boldsymbol{\nu})-E_s}{2\sigma^2}+\frac{\sqrt{E_s}\Re\{a_{2i-1}\}}{\sigma^2}\right\}. \end{aligned} \tag{38}$$

Then, by averaging with respect to the transmitted symbols, we obtain the pdf of $U_i(\boldsymbol{\nu})$ as follows:

$$P(U_i(\boldsymbol{\nu})) = \frac{1}{2\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{U_i^2(\boldsymbol{\nu})+E_s}{2\sigma^2}\right\}\cosh\left(\frac{\sqrt{E_s}}{\sigma^2}U_i(\boldsymbol{\nu})\right). \tag{39}$$

The same derivation steps can be carried out to find the pdf of $V_i(\boldsymbol{\nu})$ which is equal to the one given by (39).

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