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Closed-Form Expressions for the Exact Cramér-Rao Bounds of Timing Recovery Estimators from BPSK and Square-QAM Transmissions

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Abstract— **In this paper, we derive for the first time analytical expressions for the Cramér-Rao lower bounds (CRLBs) of timing recovery estimators from binary phase shift keying (BPSK) and square quadrature amplitude modulation (QAM) transmissions. The bounds are derived in the presence of additive white Gaussian noise (AWGN). Moreover the carrier phase and frequency are considered as unknown nuisance parameters. Our new analytical expressions reveal that the CRLBs do not depend on the corresponding time delay parameter and that they change widely from one shaping pulse to another. They also corroborate previous works that computed them empirically and provide a meaningful tool for their quick and easy evaluation.**

Index Terms— **QAM signals, symbol timing recovery, Cramér-Rao lower bound (CRLB), non-data-aided estimation.**

I. Introduction

In modern communication systems, it is often a requirement to obtain an estimate of the symbol timing. In fact, the symbol timing recovery allows sampling the signal at accurate time instants in order to achieve better performance. In practice, the symbol timing estimation depends on whether the *a priori* knowledge of the transmitted data is assumed or not. In this context, many estimators have been developed to recover the time delay of a received signal. Usually, the performance of any unbiased parameter estimator is assessed in terms of its variance, and an estimator is said to outperform another one if it exhibits lower variance. Therefore, it has been of interest to define and derive a common lower bound on the variance of unbiased estimators.

The CRLB meets this requirement and is usually used as a benchmark for the performance assessment of actual estimators [1, 2]. But due to the mathematical difficulty of the exact CRLB derivation, the modified CRLB (MCRLB) has been also introduced and derived for synchronization parameters in [3, 4]. Actually, the MCRLB is much easier to derive but, in counterpart, it departs dramatically from the exact CRLB, especially at low signal-to-noise ratio (SNR) levels. Therefore, it does not reflect the actual performance limit in this SNR region (i.e., loose bound). On the other hand, the exact CRLB is usually empirically computed using Monte-Carlo simulations. Indeed, it was empirically computed assuming the absence or the presence of unknown carrier frequency and phase in [5] and [6], respectively. In this work, we derive for the first time the exact CRLB for non-data-aided symbol timing estimation from BPSK and arbitrary square QAM-modulated signals. We assume an AWGN-corrupted received signal. Moreover, we consider the general scenario in which the carrier phase and frequency offsets are completely unknown at the receiver, and we show analytically that this assumption does not actually affect the performance of a time delay estimator from perfectly frequency and phase synchronized received samples.

The rest of this paper is organized as follows. In section II, we introduce the system model that will be used throughout this article. In sections III and IV, we derive new analytical expressions for the CRLBs in the cases of square QAM and BPSK transmissions, respectively. Some graphical representations of the newly derived bounds are presented in section V. Finally, some concluding remarks are drawn out in section VI.

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II. System Model

Consider a traditional communication system where a linearly-modulated signal is transmitted over an AWGN channel with a noise power $2\sigma^2$. Assuming imperfect frequency and phase synchronization, the received signal is modeled as follows:

$$y(t) = \sqrt{E_s}\; x(t-\tau)e^{j(2\pi f_c t+\theta)} + w(t), \tag{1}$$

where τ is the time delay to be estimated. θ is the channel distorsion phase and f_c is the carrier frequency offset. The parameters τ, θ and f_c are assumed to be deterministic but unknown. They can be gathered in the following unknown parameter vector:

$$\boldsymbol{\nu} = [\tau, \theta, f_c]^T. \tag{2}$$

Moreover, at time index, t, $w(t)$ is a complex Gaussian white noise with independent real and imaginary parts, each of variance σ^2 and $x(t)$ is the transmitted signal waveform given by:

$$x(t) = \sum_{i=1}^{N} a_i\; h(t - i\;T), \tag{3}$$

with N being the total number of transmitted symbols $\{a_i\}_{i=1}^{N}$, and T is the symbol duration. The transmitted symbols are supposed to be independent and equally likely drawn, for the time being, from any M-ary constellation where M is the modulation order. Finally, $h(t)$ is the symbol-shaping function, which will be seen, subsequently, in section III to have an important impact on the CRLB and therefore on the system performance. Suppose that we are able to produce unbiased estimates, $\widehat{\boldsymbol{\nu}}$, of the vector $\boldsymbol{\nu}$ from the received samples. Then the CRLB, which verifies $\mathrm{E}\{(\widehat{\boldsymbol{\nu}}-\boldsymbol{\nu})^2\} \geq \mathrm{CRLB}(\boldsymbol{\nu})$, is defined as [1, 2]:

$$\mathrm{CRLB}(\boldsymbol{\nu}) = \boldsymbol{I}^{-1}(\boldsymbol{\nu}), \tag{4}$$

where $\boldsymbol{I}(\boldsymbol{\nu})$ is the Fisher information matrix (FIM) whose entries are defined as:

$$[\boldsymbol{I}(\boldsymbol{\nu})]_{i;j} = E\left\{\frac{\partial L(\boldsymbol{\nu})}{\partial \nu_i}\ \frac{\partial L(\boldsymbol{\nu})}{\partial \nu_j}\right\}\ ,\ i,\ j = 1,2,3, \tag{5}$$

with $L(\boldsymbol{\nu})$ being the log-likelihood function of the parameters to be estimated and $\{\nu_i\}_{i=1}^{3}$ are the elements of $\boldsymbol{\nu}$. We also mention that $|.|$, $\Re\{.\}$, $\Im\{.\}$ and $\{.\}^*$ return the magnitude, real, imaginary and complex conjugate of any complex number and we also define the SNR of the system as $\rho = E_s/2\sigma^2$.

III. CRLB for Square QAM-modulated Signals

For the model presented in (1), the joint likelihood function can be written as [4, 7]:

$$\Lambda(y(t)|\boldsymbol{a};\boldsymbol{\nu}) = \exp\left\{\frac{\sqrt{E_s}}{\sigma^2}\int_{-\infty}^{+\infty}\Re\left\{y(t)e^{-j(2\pi f_c t+\theta)}x^*(t)\right\}dt - \frac{E_s}{2\sigma^2}\int_{-\infty}^{+\infty}|x(t)|^2 dt\right\}. \tag{6}$$

In the case of equally likely symbols, the likelihood function of the received signal is given by:

$$\Lambda\left(y(t);\tau\right) = E\left\{\prod_{i=1}^{N} F(a_i,\tilde{y}(t))\right\}, \tag{7}$$

in which:

$$F(a_i,\tilde{y}(t)) = \exp\left\{\frac{\sqrt{E_s}}{\sigma^2}\int_{-\infty}^{+\infty}\Re\{\tilde{y}(t)a_i^*\}h(t-iT-\tau)dt - \frac{E_s}{2\sigma^2}|a_i|^2\right\}, \tag{8}$$

and $\tilde{y}(t)$ is the virtually derotated received signal given by:

$$\widetilde{y}(t) = y(t)\ e^{-j(2\pi f_c t+\theta)}. \tag{9}$$

The average in (7) is performed with respect to the vector of unknown transmitted symbols. After some algebraic manipulations, it turns out that (7) reduces simply to:

$$\Lambda(y(t);\boldsymbol{\nu}) = \frac{1}{M^N}\prod_{i=1}^{N} H_i(\boldsymbol{\nu}), \tag{10}$$

with

$$H_i(\boldsymbol{\nu}) = \sum_{c_k\in C}\exp\left\{-\frac{E_s}{2\sigma^2}|c_k|^2 + \frac{\sqrt{E_s}}{\sigma^2}\int_{-\infty}^{+\infty}\Re\left\{\widetilde{y}(t)c_k^*\right\}h(t-iT-\tau)dt\right\}, \tag{11}$$

where C is the constellation alphabet. Therefore, the log-likelihood function of interest can be expressed as follows:

$$L(\boldsymbol{\nu}) = \sum_{i=1}^{N}\ln(H_i(\boldsymbol{\nu})). \tag{12}$$

Actually, even from $L(\boldsymbol{\nu})$, it is very difficult to derive a closed-form expression for the exact CRLB without elaboratiing further on $\{H_i(\boldsymbol{\nu})\}_{i=1}^{N}$ defined in (11).
Indeed, considering the special case of square QAM-modulated signals (i.e., $M = 2^{2p}$), we are able to factorize $H_i(\boldsymbol{\nu})$, which in turn makes $L(\boldsymbol{\nu})$ involve the sum of two analogous terms. In fact, denoting the alphabet of a square QAM-constellation by C, it can be readily seen that:

$$C = \tilde{C}\ \cup\ (-\tilde{C})\ \cup\ \tilde{C^*}\ \cup\ (-\tilde{C^*}), \tag{13}$$

where $\tilde{C}$ contains the symbols lying in the top-left quadrant of the constellation; i.e., $\tilde{C} = \{(2i-1)d_p + j(2k-1)d_p\}_{i,k=1,2,\ldots,2^{p-1}}$ and $2d_p$ is the inter-symbol distance which, for a normalized-energy constellation (i.e., $E\{|a_k|^2\} = 1$) is given by:

$$2d_p = 2^p/\sqrt{2^p\sum_{k=1}^{2^{p-1}}(2k-1)^2}. \tag{14}$$

Using (11) and (13) and resorting to some algebraic manipulations, $H_i(\boldsymbol{\nu})$ can be rewritten as follows:

$$H_i(\boldsymbol{\nu}) = 2\sum_{\tilde{a}_k\in\tilde{C}}\exp\left\{-\frac{E_s}{2\sigma^2}|\tilde{a}_k|^2\right\}\times \left[\cosh\left(\frac{\sqrt{E_s}}{\sigma^2}\int_{-\infty}^{+\infty}\Re\{\widetilde{y}(t)\tilde{a}_k^*\}h(t-iT-\tau)\right) + \cosh\left(\frac{\sqrt{E_s}}{\sigma^2}\int_{-\infty}^{+\infty}\Re\{\widetilde{y}(t)\tilde{a}_k\}h(t-iT-\tau)\right)\right]. \tag{15}$$

Noting that $\cosh(a)+\cosh(b) = 2\cosh(\frac{a+b}{2})\ \cosh(\frac{a-b}{2})$, it can be shown that the previous expression of $H_i(\boldsymbol{\nu})$ is factorized as follows[1]:

$$H_i(\boldsymbol{\nu}) = 4\ F(U_i(\boldsymbol{\nu}))\ F(V_i(\boldsymbol{\nu})), \tag{16}$$

where

$$U_i(\boldsymbol{\nu}) = \int_{-\infty}^{+\infty}\Re\{\widetilde{y}(t)\}\ h(t-iT-\tau)dt, \tag{17}$$

$$V_i(\boldsymbol{\nu}) = \int_{-\infty}^{+\infty}\Im\{\widetilde{y}(t)\}\ h(t-iT-\tau)dt, \tag{18}$$

and

$$F(x) = \sum_{k=1}^{2^{p-1}}\exp\left\{-\frac{E_s}{2\sigma^2}(2k-1)^2 d_p^2\right\}\cosh\left(\frac{\sqrt{E_s}}{\sigma^2}(2k-1)d_p x\right). \tag{19}$$

This factorization will be very useful to derive the analytical expressions of the exact CRLB. In fact, injecting (16) in (12), the log-lokelihood function is linearized as follows:

$$L(\boldsymbol{\nu}) = N\ln(4) + \sum_{i=1}^{N}\ln\left(F(U_i(\boldsymbol{\nu}))\right) + \sum_{i=1}^{N}\ln\left(F(V_i(\boldsymbol{\nu}))\right). \tag{20}$$

[1]Note that a similar factorization was recently used to derive an analytical expression for the NDA SNR estimates [8].

At this stage, we show in Appendix A that the problem of time delay estimation is disjoint from the problem of carrier phase and frequency estimation. Indeed, we show that the FIM is block-diagonal structured as follows:

$$\boldsymbol{I}(\boldsymbol{\nu}) = \begin{pmatrix} \text{CRLB}^{-1}(\tau) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_2(\theta, f_c) \end{pmatrix}, \quad (21)$$

where $\mathbf{0} = [0\ ,\ 0]^T$, CRLB(τ) is the CRLB of the time delay parameter and $\boldsymbol{I}_2(\theta, f_c)$ is the (2×2) FIM pertaining to the joint estimation of f_c and θ. Hence, we prove analytically, for the first time, that we deal with two separable estimation problems; on one hand time delay estimation and on the other hand carrier phase and frequency estimation. Therefore, we only need to derive the first diagonal element of the global FIM in order to derive the exact CRLB of the time delay parameter under imperfect synchronization.

The first derivative of $L(\boldsymbol{\nu})$ with respect to the time delay parameter τ is given by:

$$\frac{\partial L(\boldsymbol{\nu})}{\partial \tau} = \sum_{i=1}^{N} \frac{f(U_i(\boldsymbol{\nu}))}{F(U_i(\boldsymbol{\nu}))} \frac{\partial U_i(\boldsymbol{\nu})}{\partial \tau} + \frac{f(V_i(\boldsymbol{\nu}))}{F(V_i(\boldsymbol{\nu}))} \frac{\partial V_i(\boldsymbol{\nu})}{\partial \tau}, \quad (22)$$

where $f(x) = \frac{\partial F(x)}{\partial x}$ is given by:

$$f(x) = \sum_{k=1}^{2^{p-1}} \exp\left\{-\rho(2k-1)^2 d_p^2\right\} \times \frac{\sqrt{2\rho}}{\sigma}(2k-1)d_p \sinh\left(\frac{\sqrt{2\rho}}{\sigma}(2k-1)d_p x\right), \quad (23)$$

and ρ is the SNR at the receiver defined as $\rho = \frac{E_s}{2\sigma^2}$.

Now, we need to find the pdf of $U_i(\boldsymbol{\nu})$ and $V_i(\boldsymbol{\nu})$ before being able to derive the expectation involved in (5). To do so, we define the complex scalar random variable $Z_i(\boldsymbol{\nu}) = \int_{-\infty}^{+\infty} \widetilde{y}(t)\ h(t - iT - \tau)dt$ [note that $Z_i(\boldsymbol{\nu}) = U_i(\boldsymbol{\nu}) + jV_i(\boldsymbol{\nu})$]. Using the same algebraic manipulations from (7) through (19), it can be shown that the pdf of $Z_i(\boldsymbol{\nu})$ is given by:

$$\begin{aligned} P(Z_i(\boldsymbol{\nu});\boldsymbol{\nu}) &= \frac{1}{M}\frac{1}{2\pi\sigma^2} \exp\left\{-\frac{|Z_i(\boldsymbol{\nu})|^2}{2\sigma^2}\right\} H_i(\boldsymbol{\nu}) \\ &= \frac{4}{M2\pi\sigma^2} \exp\left\{-\frac{U_i^2(\boldsymbol{\nu}) + V_i^2(\boldsymbol{\nu})}{2\sigma^2}\right\} \times F(U_i(\boldsymbol{\nu}))F(V_i(\boldsymbol{\nu})) \\ &= P(U_i(\boldsymbol{\nu});\boldsymbol{\nu})\ P(V_i(\boldsymbol{\nu});\boldsymbol{\nu}), \end{aligned} \quad (24)$$

with

$$P(U_i(\boldsymbol{\nu});\boldsymbol{\nu}) = \frac{1}{\sqrt{2\pi\sigma^2}}\sqrt{\frac{4}{M}} \exp\left\{-\frac{U_i^2(\boldsymbol{\nu})}{2\sigma^2}\right\}\ F(U_i(\boldsymbol{\nu})), \quad (25)$$

$$P(V_i(\boldsymbol{\nu});\boldsymbol{\nu}) = \frac{1}{\sqrt{2\pi\sigma^2}}\sqrt{\frac{4}{M}} \exp\left\{-\frac{V_i^2(\boldsymbol{\nu})}{2\sigma^2}\right\}\ F(V_i(\boldsymbol{\nu})). \quad (26)$$

Denoting the first derivative of $U_i(\boldsymbol{\nu})$ with respect to τ by $\dot{U}_i(\boldsymbol{\nu})$, we have:

$$\dot{U}_i(\boldsymbol{\nu}) = \sqrt{E_s}\sum_{k=1}^{N} \Re\{a_k\}\,\dot{g}((i-k)T) - \int_{-\infty}^{+\infty} \Re\{\tilde{w}(t)\}\dot{h}(t - iT - \tau)dt, \quad (27)$$

where $\dot{g}(.)$ represents the first derivative of $g(.)$, the Nyquist pulse obtained from $h(.)$ [i.e., $g(t) = \int_{-\infty}^{+\infty} h(x)h(t+x)dx$]. Then, it is easy to verify that:

$$\begin{aligned} \mathrm{E}\left\{\left(\frac{\partial L(\boldsymbol{\nu})}{\partial \tau}\right)^2\right\} &= \mathrm{E}\left\{\left(\sum_{i=1}^{N} \frac{f(U_i(\boldsymbol{\nu}))}{F(U_i(\boldsymbol{\nu}))}\ \dot{U}_i(\boldsymbol{\nu})\right)^2\right\} \\ &= \mathrm{E}\left\{\left(\sum_{i=1}^{N} \frac{f(V_i(\boldsymbol{\nu}))}{F(V_i(\boldsymbol{\nu}))}\ \dot{V}_i(\boldsymbol{\nu})\right)^2\right\} \\ &= 2\sum_{i=1}^{N}\sum_{l=1}^{N} E\left\{\frac{f(U_i(\boldsymbol{\nu}))}{F(U_i(\boldsymbol{\nu}))}\ \dot{U}_i(\boldsymbol{\nu}) \frac{f(U_l(\boldsymbol{\nu}))}{F(U_l(\boldsymbol{\nu}))}\ \dot{U}_l(\boldsymbol{\nu})\right\}, \end{aligned} \quad (28)$$

where the last equality follows from the fact that $U_i(\boldsymbol{\nu})$ and $V_i(\boldsymbol{\nu})$ are identically distributed according to (25) and (26). For a further development, it is more convenient to discuss the case where $i = l$ separately from the case where $i \neq l$.

Considering $i = l$, we show in Appendix B that $U_i(\boldsymbol{\nu})$ and $\dot{U}_i(\boldsymbol{\nu})$ are independent. Therefore, we split the expectations and we obtain:

$$E\left\{\left(\frac{f(U_i(\boldsymbol{\nu}))}{F(U_i(\tau))}\right)^2 (\dot{U}_i(\tau))^2\right\} = E\left\{\left(\frac{f(U_i(\boldsymbol{\nu}))}{F(U_i(\tau))}\right)^2\right\} \times E\left\{\left(\dot{U}_i(\tau)\right)^2\right\}. \quad (29)$$

where $f(.)$ and $F(.)$ are defined in (23) and (19), respectively. By using the distribution of $U_i(\boldsymbol{\nu})$ defined in (25), we obtain:

$$\mathrm{E}\left\{\left(\frac{f(U_i(\boldsymbol{\nu}))}{F(U_i(\boldsymbol{\nu}))}\right)^2\right\} = \sqrt{\frac{4}{M}}\frac{1}{\sqrt{2\pi}\ \sigma} \times \int_{-\infty}^{+\infty} \frac{f^2(U_i(\boldsymbol{\nu}))}{F(U_i(\boldsymbol{\nu}))} \exp\left\{-\frac{U_i^2(\boldsymbol{\nu})}{2\sigma^2}\right\}\ dU_i(\boldsymbol{\nu}), \quad (30)$$

and it is easy to see that

$$\mathrm{E}\left\{\left(\dot{U}_i(\boldsymbol{\nu})\right)^2\right\} = \frac{E_s}{2}\sum_{k=1}^{N} \dot{g}^2((i-k)T) - \sigma^2\ddot{g}(0) \quad (31)$$

where $\ddot{g}(.)$ is the second derivative of $g(.)$.

Now, in the case where $i \neq l$, developments are more complicated. The intersymbol interference results in a statistical dependence between $U_i(\boldsymbol{\nu})$, $\dot{U}_i(\boldsymbol{\nu})$ and $\dot{U}_l(\boldsymbol{\nu})$. Therefore, we adopt an other approach to compute the considered expectation. We first average by conditioning on $U_i(\boldsymbol{\nu})$ and $U_l(\boldsymbol{\nu})$ then, average the result with respect to these two variables. So, let us start by considering the expectation of $\dot{U}_i(\boldsymbol{\nu})$ conditioned on $U_i(\boldsymbol{\nu})$ and $U_l(\boldsymbol{\nu})$:

$$E\{\dot{U}_i(\boldsymbol{\nu})|U_i(\boldsymbol{\nu}), U_l(\boldsymbol{\nu})\} = U_l(\boldsymbol{\nu})\dot{g}((i-l)T). \quad (32)$$

It follows that:

$$E\left\{\frac{f(U_i(\boldsymbol{\nu}))}{F(U_i(\boldsymbol{\nu}))}\frac{f(U_l(\boldsymbol{\nu}))}{F(U_l(\boldsymbol{\nu}))}\,\dot{U}_i(\boldsymbol{\nu})\dot{U}_l(\boldsymbol{\nu})\Big|U_i(\boldsymbol{\nu}),U_l(\boldsymbol{\nu})\right\}=$$
$$\frac{f(U_i(\boldsymbol{\nu}))}{F(U_i(\boldsymbol{\nu}))}\frac{f(U_l(\boldsymbol{\nu}))}{F(U_l(\boldsymbol{\nu}))}U_i(\boldsymbol{\nu})U_l(\boldsymbol{\nu})\dot{g}((i-l)T)\dot{g}((l-i)T). \quad (33)$$

Exploiting the fact that $U_i(\boldsymbol{\nu})$ and $U_i(\boldsymbol{\nu})$ are statisticaly independent, we obtain:

$$E\left\{\frac{f(U_i(\boldsymbol{\nu}))}{F(U_i(\boldsymbol{\nu}))}\frac{f(U_l(\boldsymbol{\nu}))}{F(U_l(\boldsymbol{\nu}))}\,\dot{U}_i(\boldsymbol{\nu})\dot{U}_l(\boldsymbol{\nu})\right\}=$$
$$-\left(E\left\{\frac{f(U_i(\boldsymbol{\nu}))}{F(U_i(\boldsymbol{\nu}))}U_i(\boldsymbol{\nu})\right\}\right)^2\dot{g}^2((i-l)T). \quad (34)$$

Finally, gathering the expressions in (30) and (31), and using the pdf of $U_i(\boldsymbol{\nu})$ to compute the expectation in the right-hand side of (34), the analytical expression of the CRLB for timing estimation is given by the following formula:

$$\text{CRLB}(\tau)=\left[\left(2\rho^2\sum_{m=1}^{N}\sum_{n=1}^{N}\dot{g}^2((m-n)T)-2N\rho\,\ddot{g}(0)\right)\times\right.$$
$$\sqrt{\frac{2}{\pi M}}\int_{-\infty}^{+\infty}\frac{g_\rho^2(x)}{G_\rho(x)}e^{-\frac{x^2}{2}}dx-\frac{4\rho}{\pi M}\left(\int_{-\infty}^{+\infty}xg_\rho(x)e^{-\frac{x^2}{2}}dx\right)^2\times$$
$$\left.\sum_{m=1}^{N}\sum_{n=1}^{N}\dot{g}^2((m-n)T)\right]^{-1}, \quad (35)$$

where some simplifications are made by changing $\frac{U_i(\boldsymbol{\nu})}{\sigma}$ by x and the two functions $g_\rho(.)$ and $G_\rho(.)$ are defined as:

$$g_\rho(x)=\sum_{k=1}^{2^{p-1}}\exp\left\{-\rho(2k-1)^2d_p^2\right\}\sqrt{2}(2k-1)d_p\times$$
$$\sinh\left(\sqrt{2\rho}(2k-1)d_px\right), \quad (36)$$

and

$$G_\rho(x)=\sum_{k=1}^{2^{p-1}}\exp\left\{-\rho(2k-1)^2d_p^2\right\}\times$$
$$\cosh\left(\sqrt{2\rho}(2k-1)d_px\right). \quad (37)$$

Note that when the observation interval NT is longer than the duration of $\dot{g}(.)$, one can use the following approximation [5]:

$$\sum_{m=1}^{N}\sum_{n=1}^{N}\dot{g}^2((m-n)T)\approx N\sum_{m=-\infty}^{+\infty}\dot{g}^2(mT). \quad (38)$$

We mention here that the new analitical expression of the CRLB given in (35) allows immediate evaluation of the time delay stochastic CRLBs for all QAM modulation orders. Besides, the analitical expression reveals that the CRLB is a function of the SNR, the modulation order and the shaping pulse function. In addition, our new analytical expression shows that the CRLB does not depend on the corresponding time delay parameter, as expected intuitively.

IV. CRLB FOR BPSK-MODULATED SIGNALS

In the case of BPSK signaling, the noise is real valued of power σ^2. In this case, it can be easily shown that the log-likelihood function of $y(t)$ is given by:

$$L(\boldsymbol{\nu})=\sum_{i=1}^{N}\ln\left(\frac{1}{2}\exp\left\{\frac{\sqrt{E_s}}{\sigma^2}\int_{-\infty}^{+\infty}y(t)h(t-iT-\tau)dt\right\}\right.$$
$$\left.+\exp\left\{-\frac{\sqrt{E_s}}{\sigma^2}\int_{-\infty}^{+\infty}y(t)h(t-iT-\tau)dt\right\}\right)$$
$$=\sum_{i=1}^{N}\ln\left(\cosh\left(\frac{\sqrt{E_s}}{\sigma^2}\int_{-\infty}^{+\infty}y(t)\times\right.\right.$$
$$\left.\left.h(t-iT-\tau)dt\right)\right). \quad (39)$$

The first derivative of $L(\boldsymbol{\nu})$ with respect to the time delay parameter τ is given by:

$$\frac{\partial L(\boldsymbol{\nu})}{\partial\tau}=\frac{\sqrt{E_s}}{\sigma^2}\sum_{i=1}^{N}\frac{\sinh\left(\frac{\sqrt{E_s}}{\sigma^2}U_i(\boldsymbol{\nu})\right)}{\cosh\left(\frac{\sqrt{E_s}}{\sigma^2}U_i(\boldsymbol{\nu})\right)}\frac{\partial U_i(\boldsymbol{\nu})}{\partial\tau}, \quad (40)$$

where

$$U_i(\boldsymbol{\nu})=\int_{-\infty}^{+\infty}y(t)\;h(t-iT-\tau)dt. \quad (41)$$

Starting from (40) and following the same steps of section III, we show that the CRLB for a BPSK modulation is given by[2]:

$$\text{CRLB}=\frac{1}{4\rho}\left[\left(1-\frac{1}{\sqrt{2\pi}}e^{-\rho}\beta(\rho)\right)\left(\rho\sum_{m=1}^{N}\sum_{n=1}^{N}\dot{g}((m-n)T)-\right.\right.$$
$$\left.\left.\frac{N}{2}\ddot{g}(0)\right)-\rho\sum_{m=1}^{N}\sum_{n=1}^{N}\dot{g}((m-n)T)\right]^{-1}, \quad (42)$$

where $\beta(.)$ is defined as:

$$\beta(\rho)=\int_{-\infty}^{+\infty}\frac{e^{-\frac{x^2}{2}}}{\cosh\left(\sqrt{2\rho}\;x\right)}dx. \quad (43)$$

More delays about the derivation are presented in [7]. It is worth noting that the factorization of the log-likelihood function for a BPSK signal is straightforward compared to the case of square QAM-modulated signals.

V. GRAPHICAL REPRESENTATION

Fig. 1 depicts the CRLB / MCRLB ratio for BPSK and square-QAM modulations. This ratio informs about the performance degradation that stems from randomizing the transmitted sequence. It converges to one at sufficiently high SNR values and it increases dramatically in the low SNR levels. This means that in the latter SNR region, the *a priori* knowledge of the transmitted data is compulsory if reliable estimates of the time delay parameter are required. We consider in this figure two values of the roll-off factor, 0.2 and 1, in order to show that our analytical expressions corroborate previous

[2]Note that the SNR for a BPSK signal is E_s/σ^2 instead of $E_s/2\sigma^2$ in the case of square QAM transmissions.

attemps to empirically evaluate the same CRLBs in [5]. In fact, we see a good agrement. Clearly, the CRLB decreases with increasing the roll-off factor, and this is due to the decrease of the intersymbol interference.

Moreover, it can be seen from Fig. 2 that, for high SNR values, all the CRLBs converge to the MCRLB which itself can be shown to coincide with the data-aided CRLB. Therefore, in this SNR region, the achievable performance is equivalent to the one that can be obtained if the transmitted symbols were assumed to be perfectly known.

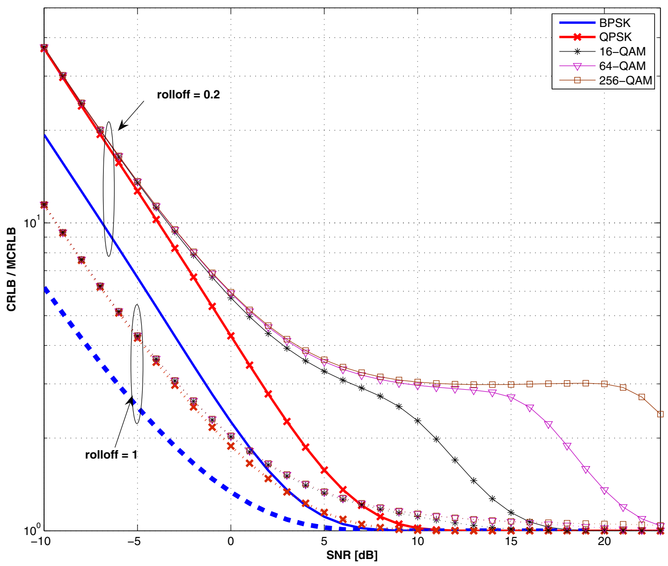

Fig. 1. CRLB/MCRLB ratio vs. SNR for different modulation orders using $K = 100$ and a raised cosine pulse with rolloff factor of 0.2 and 1.

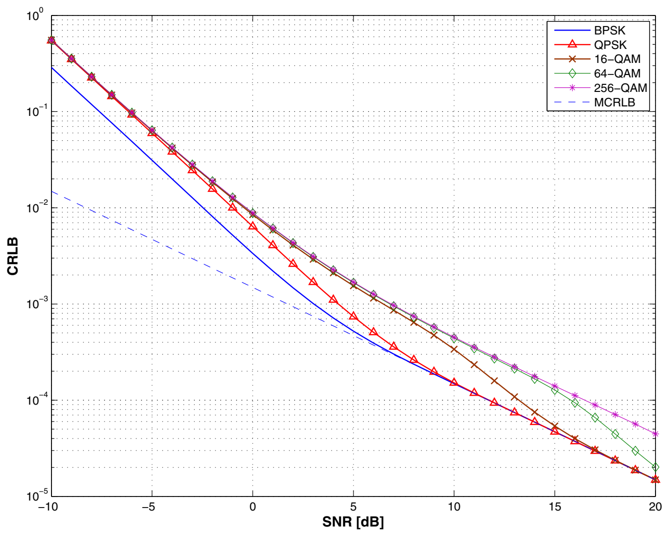

Fig. 2. CRLB vs. SNR for different modulation orders using $K = 100$ and a raised cosine pulse with rolloff factor of 0.2.

VI. Conclusion

In this paper, we derived for the first time analytical expressions of the exact Cramér-Rao lower bound for symbol timing estimation in the case of BPSK and square-QAM modulations. We focused on the case of stochastic CRLB (unknown transmitted data) and we showed that the bound does not depend on the true time delay value. The carrier phase and frequency offsets were also unknown (nuisance parameters). First, we showed that the knowledge of these two nuisance parameters does not bring any additional information to time delay estimation since the latter is decoupled from the joint estimation of the carrier frequency and phase offsets. Moreover, our analytical expressions corroborate previous attempts to empirically compute the considered CRLBs. Finally, the derived expressions offer an efficient tool for a quick and easy evaluation of the NDA CRLBs for BPSK and arbitrary square-QAM transmissions so timely for current and future communication systems that are expected to operate with high data rates.

Appendix A
Proof of the Block-Diagonal Structure of the FIM

To show that τ and $\boldsymbol{u} = [f_c, \theta]^T$ are decoupled, we consider the formulation in (20) of the log-likelihood function. The derivatives of this function with respect to $\boldsymbol{u}(l)$, $l = 1, 2$ and τ are, respectively, given by:

$$\frac{\partial L(\boldsymbol{\nu})}{\partial \boldsymbol{u}(l)} = \sum_{i=1}^{N} \frac{f(U_i(\boldsymbol{\nu}))}{F(U_i(\boldsymbol{\nu}))} \frac{\partial U_i(\boldsymbol{\nu})}{\partial \boldsymbol{u}(l)} + \frac{f(V_i(\boldsymbol{\nu}))}{F(V_i(\boldsymbol{\nu}))} \frac{\partial V_i(\boldsymbol{\nu})}{\partial \boldsymbol{u}(l)}, \; l = 1, 2, \tag{44}$$

and

$$\frac{\partial L(\boldsymbol{\nu})}{\partial \tau} = \sum_{i=1}^{N} \frac{f(U_i(\boldsymbol{\nu}))}{F(U_i(\boldsymbol{\nu}))} \frac{\partial U_i(\boldsymbol{\nu})}{\partial \tau} + \frac{f(V_i(\boldsymbol{\nu}))}{F(V_i(\boldsymbol{\nu}))} \frac{\partial V_i(\boldsymbol{\nu})}{\partial \tau}, \tag{45}$$

where $f(.)$ is defined in (23). Then we average $\frac{\partial L(\boldsymbol{\nu})}{\partial \tau}\frac{\partial L(\boldsymbol{\nu})}{\partial \boldsymbol{u}(l)}$ as in (28) to obtain the following result:

$$\begin{aligned}[\boldsymbol{I}(\boldsymbol{\nu})]_{1;l+1} =& \mathrm{E}\left\{\frac{\partial L(\boldsymbol{\nu})}{\partial \tau}\frac{\partial L(\boldsymbol{\nu})}{\partial \boldsymbol{u}(l)}\right\} \\ =& \, 2\sum_{i=1}^{N}\sum_{m=1}^{N} E\left\{\frac{\dot{F}(U_i(\boldsymbol{\nu}))}{F(U_i(\boldsymbol{\nu}))}\frac{\dot{F}(U_m(\boldsymbol{\nu}))}{F(U_m(\boldsymbol{\nu}))}\right. \\ & \left.\frac{\partial U_i(\boldsymbol{\nu})}{\partial \tau}\frac{\partial U_m(\boldsymbol{\nu})}{\partial \boldsymbol{u}(l)}\right\}. \end{aligned} \tag{46}$$

In order to simplify the calculations, without loss of generality, we consider $l = 1$. To begin with, we first differentiate $U_m(\boldsymbol{\nu})$ with respect to f_c and we obtain:

$$\begin{aligned}\frac{\partial U_m(\boldsymbol{\nu})}{\partial f_c} &= 2\pi \int_{-\infty}^{+\infty} \Im\left\{y(t)e^{-j(2\pi f_c t + \theta)}\right\} h(t - mT - \tau)tdt \\ &= 2\pi \sum_{n=1}^{N} \Im\{a_m\} \int_{-\infty}^{+\infty} h(t - nT - \tau) \times \\ & h(t - mT - \tau)tdt + \int_{-\infty}^{+\infty} \Im\{\widetilde{w}(t)\} h(t - mT - \tau)tdt. \end{aligned} \tag{47}$$

Obviously, $\frac{\partial U_m(\boldsymbol{\nu})}{\partial f_c}$ is function of the imaginary part of the derotated received signal, which is independent from its real part. Therefore, $\frac{\partial U_m(\boldsymbol{\nu})}{\partial f_c}$ is independent from $U_i(\boldsymbol{\nu})$, $U_m(\boldsymbol{\nu})$ and $\frac{\partial U_i(\boldsymbol{\nu})}{\partial \tau}$. As a result, we can split the expectation in (46) as follows:

$$E\left\{\frac{\dot{F}(U_i(\boldsymbol{\nu}))}{F(U_i(\boldsymbol{\nu}))}\frac{\dot{F}(U_m(\boldsymbol{\nu}))}{F(U_m(\boldsymbol{\nu}))}\frac{\partial U_i(\boldsymbol{\nu})}{\partial \tau}\frac{\partial U_m(\boldsymbol{\nu})}{\partial u_l}\right\} = E\left\{\frac{\dot{F}(U_i(\boldsymbol{\nu}))}{F(U_i(\boldsymbol{\nu}))}\frac{\dot{F}(U_m(\boldsymbol{\nu}))}{F(U_m(\boldsymbol{\nu}))}\frac{\partial U_i(\boldsymbol{\nu})}{\partial \tau}\right\} E\left\{\frac{\partial U_m(\boldsymbol{\nu})}{\partial u_l}\right\}. \quad (48)$$

And noting that the last expectation is equal to zero, it follows that the two parameters τ and f_c are decoupled. The same manipulations are used to prove that τ and θ are also decoupled. Therefore, the FIM is block-diagonal structured as given by (21).

Appendix B

Proof of the independence of $U_i(\boldsymbol{\nu})$ and $\dot{U}_i(\boldsymbol{\nu})$

In this Appendix, we show the independence between $U_i(\boldsymbol{\nu})$ and $\dot{U}_i(\boldsymbol{\nu})$. First, $U_i(\boldsymbol{\nu})$ can be written as:

$$U_i(\boldsymbol{\nu}) = \alpha_i + \beta_i, \quad (49)$$

where

$$\alpha_i = \sqrt{E_s}\Re\{a_i\}, \quad (50)$$

and

$$\beta_i = \int_{-\infty}^{+\infty} \Re\{\tilde{w}(t)\}h(t - iT - \tau)dt. \quad (51)$$

Therefore, $\dot{U}_i(\boldsymbol{\nu})$ is given by:

$$\begin{aligned}\dot{U}_i(\tau) &= \sum_{m=1}^{N} \Re\{a_m\}\dot{g}((i-m)T) - \int_{-\infty}^{+\infty} \Re\{\widetilde{w}(t)\}\dot{h}(t - iT - \tau)dt \\ &= \sum_{m=1}^{N} \Re\{a_m\}\dot{g}((i-m)T) - \dot{\beta}_i. \quad (52)\end{aligned}$$

α_i and $\dot{\beta}_i$ are independent since the noise and the transmitted symbol are independent. Moreover, β_i and $\dot{U}_i(\boldsymbol{\nu})$ are obtained by a linear transformation of the Gaussian process $\Re\{\tilde{w}(t)\}$. Hence they are also Gaussian processes. Moreover, the cross-correlation of β_i and $\dot{\beta}_i$ is equal to zero, as shown below:

$$\begin{aligned}\mathrm{E}\{\beta_i\dot{\beta}_i\} = {} & \mathrm{E}\left\{\iint_{-\infty}^{+\infty} \Re\{w(t_1)e^{-j(2\pi f_c t_1+\theta)}\} \times \Re\{w(t_2)e^{-j(2\pi f_c t_2+\theta)}\}h(t_1 - iT - \tau)\dot{h}(t_2 - iT - \tau)dt_1dt_2\right\} \\ &= \frac{\sigma^2}{2}\iint_{-\infty}^{+\infty} \delta(t_1 - t_2)h(t_1)\dot{h}(t_2)dt_1dt_2 \\ &= \frac{\sigma^2}{2}\dot{g}(0) \\ &= 0, \quad (53)\end{aligned}$$

where the last equality follows from the fact that the Nyquist pulse is maximum in 0 and therefore $\dot{g}(0) = 0$. And recall that $\dot{g}(0) = 0$, then $\sum_{m=1}^{N} \Re\{a_m\}\dot{g}((i-m)T)$ and α_i are independent. We conclude that $U_i(\boldsymbol{\nu})$ and $\dot{U}_i(\boldsymbol{\nu})$ are also independent.

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