A New OFDM Synchronization Symbol for Carrier Frequency Offset Estimation

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***Abstract*—This letter proposes a new data-aided carrier frequency offset (CFO) estimation scheme for orthogonal frequency division multiplexing (OFDM) communications suitable for frequency-selective channels. The proposed method is based on the transmission of a specially designed synchronization symbol that generates a particular signal structure between the received observation samples at the receiver. This structure is exploited to derive a closed-form expression of the CFO. The proposed solution offers a wide acquisition range with reduced computational load. Simulations over frequency-selective channels confirm the superiority of the proposed method compared to recent data-aided synchronization algorithms.**

***Index Terms*—Frequency-division multiplexing, synchronization.**

I. Introduction

ORTHOGONAL frequency division multiplexing (OFDM) performance suffers from a pronounced sensitivity against carrier frequency offsets (CFOs) [1]. The CFO should therefore be estimated and compensated before any further processing in the receiver. Different data-aided schemes of CFO estimation in OFDM systems have been appraised. These methods hinge on the periodic transmission of known data blocks that allows for the estimation of the CFO from the estimation of the phase rotation between these blocks at the receiver. The most known of these methods is the M&M method [3], which constitutes an improvement to Schmidl's [2]. The M&M method was also selected by [4] to provide a joint robust estimation of the timing and the frequency offset. More recently, a new synchronization symbol proposed in [5] allowed for performance gains over Schmidl's algorithm.

In this letter, we propose a new synchronization symbol structure and a new CFO estimator that offer a wide acquisition range and a high accuracy at a very reduced computational cost. Based on the original shape of the new synchronization symbol, a closed-form CFO estimator is derived. Our method has an acquisition range that can reach $\pm N/2p$ the subcarrier spacing, where p is a user-selected parameter characterizing our synchronization symbol, and N is the number of OFDM subcarriers. As supported by simulations, our algorithm provides higher accuracy compared to the well-known M&M method [3] as well as the method in [5]. More specifically, our estimator proved to be robust against low SNRs (below 0 dB). Such low SNR values are of great practical interest since next-generation wireless systems attempt to operate in MIMO multi-access multi-cell environments, which are characterized by low operating SNRs. In this regard, OFDM needs reliable synchronization techniques that operate well in a low SNR regime.

This letter is organized as follows. In Section II, the OFDM system model is described. The designed synchronization symbol and the resulting structure in the received samples are described in Section III. The CFO estimator and the different derivations are presented in Section IV. Section V analyzes the performance of the proposed estimator. Numerical examples are illustrated in Section VI. Conclusions are given in Section VII.

Manuscript received July 4, 2006; revised September 21, 2006. This work was supported by a Canada Research Chair in High-Speed Wireless Communications. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Gerald Matz.

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Digital Object Identifier 10.1109/LSP.2006.888353

II. System Model

We consider a discrete-time OFDM system with N subcarriers. At the transmitter, N complex-valued symbols X_k, which belong to a QAM or PSK constellation, modulate N orthogonal subcarriers using the inverse fast Fourier transform (IFFT). Before transmission, a cyclic prefix is appended at the beginning of the signal, yielding

$$s(n) = \begin{cases} x(n+N), & \text{if } -N_g \le n < 0 \\ x(n), & \text{if } 0 \le n \le N-1 \end{cases}$$

where $x(n) = (1/N)\sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}, n = 0, \ldots, N-1$.

At the receiver, the cyclic prefix is discarded, leading to the following received samples:

$$r(n) = e^{j2\pi n\varepsilon/N} \sum_{l=0}^{N_c-1} h_l s(n-l) + \mu(n) \quad (1)$$

$$= \frac{e^{j2\pi n\varepsilon/N}}{N} \sum_{k=0}^{N-1} X_k H_k e^{j2\pi kn/N} + \mu(n) \quad (2)$$

$$= y(n) + \mu(n), \quad n = 0, \ldots, N-1 \quad (3)$$

where $H_k = \sum_{l=0}^{N_c-1} h_l e^{-j2\pi kl/N}$ is the transfer function of the channel at the frequency of the kth subcarrier, N_c corresponds to the channel length, ε is the relative carrier frequency offset (the ratio of the actual frequency offset to the intercarrier spacing), and $\mu(n)$ is an additive complex white Gaussian noise (with variance σ_μ^2) independent of $y(n)$.

III. Synchronization Symbol Design

To estimate the CFO, we propose to transmit the following synchronization symbol:

$$X_{k+1} = e^{j2\pi pk/N} X_k, \quad k = 0, \ldots, N-2 \quad (4)$$

or equivalently, for $k \in \{1, \ldots, N-1\}$

$$X_k = \exp\left(\frac{j2\pi p \sum_{i=0}^{k-1} i}{N}\right) X_0 = e^{j\pi(pk(k-1)/N)} X_0 \tag{5}$$

where p is a user-selected even integer such that $p \in \{2, \ldots, N-2\}$. The choice of p has to be taken under certain considerations such as the desired acquisition range, the desired accuracy, and the PAPR that can be tolerated by the system.

Based on the special structure of this synchronization symbol, we will prove that there is a useful relationship between $r(n)$ and $r(n+p)$ that will allow us to easily derive a closed-form estimator of the CFO. First, let us consider the noiseless part of $r(n+p)$ in the following:

$$y(n+p) = \frac{e^{j2\pi(n+p)\varepsilon/N}}{N} \sum_{k=0}^{N-1} X_k H_k e^{j2\pi k(n+p)/N} \tag{6}$$

$$= \frac{e^{j2\pi(n+p)\varepsilon/N}}{N} \left[\sum_{k=0}^{N-2} H_k e^{j2\pi kn/N} X_{k+1} + H_{N-1} e^{j2\pi(N-1)n/N} X_{N-1} e^{j2\pi p(N-1)/N}\right] \tag{8}$$

$$= \frac{e^{j2\pi[(n+p)\varepsilon-n]/N}}{N} \left[\sum_{k=1}^{N-1} H_{k-1} e^{j2\pi kn/N} X_k + H_{N-1} X_{N-1} e^{j2\pi p(N-1)/N}\right]. \tag{10}$$

In a system with many subcarriers, the inequality $N_c \ll N$ is always fulfilled. In this case, $H_k \approx H_{k-1}$ for $k = 1, \ldots, N-1$, which is equivalent to saying that adjacent subcarriers experience approximately the same channel. Moreover, due to the very form of H_k and since $1 \leq N_c \ll N$, it can be easily shown that $H_0 \approx H_{N-1}$. Therefore

$$y(n+p) \simeq e^{j2\pi[(n+p)\varepsilon-n]/N} \frac{1}{N} \left[\sum_{k=1}^{N-1} H_k e^{j2\pi kn/N} X_k + H_0 X_{N-1} e^{j2\pi p(N-1)/N}\right]. \tag{11}$$

The second term $X_{N-1} e^{j2\pi p(N-1)/N}$ could be further developed [using (5)]

$$X_{N-1} e^{j2\pi(p(N-1)/N)} = X_0 e^{j\pi p(N-1)}. \tag{12}$$

Since p is even, then $X_{N-1} e^{j2\pi p(N-1)/N} = X_0$. Substituting this last equation in (11), we obtain:

$$y(n+p) \simeq e^{j2\pi[p\varepsilon-n]/N} y(n). \tag{13}$$

Based on this last equation, we can now state that

$$r(n+p) \approx e^{j2\pi[\varepsilon p-n]/N} y(n) + \mu(n+p). \tag{14}$$

IV. Proposed CFO Estimator

The proposed CFO estimator is based on the correlation between the received samples spaced by p lags. Let us consider the following correlation product:

$$e^{j2\pi n/N}[y(n+p)y^*(n) + y(n+p)\mu^*(n) + \mu(n+p)y^*(n) + \mu(n+p)\mu^*(n)]. \tag{16}$$

For high SNR (SNR $= E[|y(n)|^2]/\sigma_\mu^2$), we may neglect the term $\mu(n+p)\mu^*(n)$, and by using (13), we have that

$$r(n+p)r^*(n)e^{j2\pi n/N} = e^{j2\pi\varepsilon p/N}(|y(n)|^2 + v(n+p) + v^*(n)) \tag{18}$$

where $v(n) = y^*(n)\mu(n)$. We therefore propose the following CFO estimator:

$$\hat{\varepsilon} = \frac{N}{2\pi p} \angle \left[\sum_{n=0}^{N-p-1} w(n) r(n+p) r^*(n) e^{j2\pi n/N}\right] \tag{19}$$

where $w(n), n = 0 \ldots N-p-1$ is a window of real weights designed to minimize the estimation variance and such that $\sum_{n=0}^{N-p-1} w(n) = 1$. One can notice that this estimator will induce a very small computational load. Indeed, it involves a unique phase angle calculation, and its very form indicates that it can be implemented efficiently using an FFT, a component that already exists in the OFDM receiver. Moreover, the acquisition range of this estimator is $N/2p$, which can be large if one chooses a low value of p. In the section that follows, we will derive closed-form expressions of both the window $w(n)$ and the theoretical variance bounds of our estimator.

V. Performance Analysis

Inserting (18) in (19), we obtain that

$$\hat{\varepsilon} = \varepsilon + \frac{N}{2\pi p} \angle \left[\sum_{n=0}^{N-p-1} w(n)(|y(n)|^2 + v(n+p) + v^*(n))\right]$$

or in other words, if $v(n) = v_1(n) + jv_2(n)$, we have that

$$\hat{\varepsilon} - \varepsilon \simeq \frac{N}{2\pi p} \angle [A + X + jY] = \frac{N}{2\pi p} \angle \left[1 + \frac{X}{A} + j\frac{Y}{A}\right] \tag{20}$$

where $A = \sum_{n=0}^{N-p-1} w(n)|y(n)|^2$

$$X = \sum_{n=0}^{N-p-1} w(n)(v_1(n+p) + v_1(n)) \tag{21}$$

and $Y = \sum_{n=0}^{N-p-1} w(n)(v_2(n+p) - v_2(n))$.

Since $\sum_{n=0}^{N-p-1} w(n) = 1$, we may approximate A by

$$A = \sum_{n=0}^{N-p-1} w(n)|y(n)|^2 \approx E[|y(n)|^2]. \tag{22}$$

Therefore, we have

$$\hat{\varepsilon} - \varepsilon \approx \frac{N}{2\pi p} \angle \left[1 + \frac{X}{E[|y(n)|^2]} + j\frac{Y}{E[|y(n)|^2]}\right]. \tag{23}$$

Since

$$\begin{cases} E[v_1(n)] = E[v_2(n)] = 0 \\ E[v_1(n)v_1(k)] = E[v_2(n)v_2(k)] = \frac{E[v(n)v^*(k)]}{2} \\ = \frac{\delta_{n,k} E[|y(n)|^2]\sigma_\mu^2}{2}, \text{ where } \delta_{n,k} \text{ is the Kronecker } \delta \text{ function} \end{cases}$$

it is easy to see that $E[X] = E[Y] = 0$ and that

$$E[Y^2] = \sigma_\mu^2 E[|y(n)|^2]\alpha \tag{24}$$

where

$$\alpha = \sum_{k=0}^{N-p-1}\sum_{n=0}^{N-p-1} w(k)w(n)\left(\delta_{n,k} - \frac{1}{2}\delta_{n,k+p} - \frac{1}{2}\delta_{n+p,k}\right). \tag{25}$$

Similarly, $E[X^2] = \sigma_\mu^2 E[|y(n)|^2]\beta$, where

$$\beta = \sum_{k=0}^{N-p-1}\sum_{n=0}^{N-p-1} w(k)w(n)\left(\delta_{n,k} + \frac{1}{2}\delta_{n,k+p} + \frac{1}{2}\delta_{n+p,k}\right). \tag{26}$$

From the Central Limit Theorem (CLT), we have that $Y/E[|y(n)|^2] \sim \mathcal{N}(0, \alpha/SNR)$ and $X/E[|y(n)|^2] \sim \mathcal{N}(0, \beta/SNR)$. For high SNR, we may consequently state that [6]

$$\begin{aligned} \angle\left[1 + \frac{X}{E[|y(n)|^2]} + j\frac{Y}{E[|y(n)|^2]}\right] &= \angle\left[1 + x + jy\right] \\ = \angle\left[\exp\left(j\tan^{-1}\left(\frac{y}{1+x}\right)\right)\right] &\approx \angle\left[\exp(j\tan^{-1}(y))\right] \\ \approx \angle\left[\exp(jy)\right] = y = \frac{Y}{E[|y(n)|^2]}. \end{aligned} \tag{27}$$

Consequently, we have

$$\hat{\varepsilon} - \varepsilon \approx \frac{N}{2\pi p}\frac{Y}{E[|y(n)|^2]}. \tag{28}$$

Therefore

$$\begin{cases} E[\hat{\varepsilon}] = \varepsilon, \text{ i.e., the estimator is unbiased for high SNR} \\ E[(\hat{\varepsilon} - \varepsilon)^2] = \frac{N^2}{8\pi^2 p^2}\frac{2\alpha}{SNR}. \end{cases}$$

The variance of the proposed CFO estimator can be rewritten as

$$E[(\hat{\varepsilon} - \varepsilon)^2] = \frac{N^2}{8\pi^2 p^2}\frac{1}{SNR}\mathbf{w}\boldsymbol{\Sigma}\mathbf{w}^T \tag{29}$$

where $\mathbf{w} = [w(0), \ldots, w(N-p-1)]$, and $\boldsymbol{\Sigma}$ is an $(N-p) \times (N-p)$ matrix with entries equal to $\sigma(i,l) = 2\delta_{i,l} - \delta_{i,l+p} - \delta_{i+p,l}$. Following [7], the window that minimizes the variance is such that

$$\mathbf{w} = \frac{\mathbf{i}_{N-p}\boldsymbol{\Sigma}^{-1}}{\mathbf{i}_{N-p}\boldsymbol{\Sigma}^{-1}\mathbf{i}_{N-p}^T} \tag{30}$$

where $\mathbf{i}_{N-p}$ stands for an $(N-p)$-dimensional row vector consisting of only ones. The corresponding estimation variance is

$$E[(\hat{\varepsilon} - \varepsilon)^2] = \frac{N^2}{8\pi^2 p^2 SNR}\frac{1}{\mathbf{i}_{N-p}\boldsymbol{\Sigma}^{-1}\mathbf{i}_{N-p}^T}. \tag{31}$$

Explicit forms of the variance and the optimal window were discussed extensively in [7]. Here, we just briefly cite some important results.

1) For $p \geq N/2$, we have that $\boldsymbol{\Sigma} = 2\mathbf{I}$, and therefore

$$\begin{cases} \mathbf{w} = \frac{1}{N-p}\mathbf{i}_{N-p} \\ E[(\hat{\varepsilon} - \varepsilon)^2] = \frac{N^2}{4\pi^2 p^2 SNR(N-p)}. \end{cases}$$

2) For $\lceil N/3 \rceil \leq p \leq N/2$, the inverse of $\boldsymbol{\Sigma}$ is such as

$$\boldsymbol{\Sigma}^{-1} = \frac{1}{3}\begin{bmatrix} 2\mathbf{I}_{N-2p} & \mathbf{0} & \mathbf{I}_{N-2p} \\ \mathbf{0} & \frac{3}{2}\mathbf{I}_{3p-N} & \mathbf{0} \\ \mathbf{I}_{N-2p} & \mathbf{0} & 2\mathbf{I}_{(N-2p)} \end{bmatrix}. \tag{32}$$

The corresponding optimal window is given by

$$w(k) = \begin{cases} \frac{2}{3N-5p}, & \text{for } k = 0, \ldots N-2p-1 \\ \frac{1}{3N-5p}, & \text{for } k = N-2p, \ldots p-1 \\ \frac{2}{3N-5p}, & \text{for } k = p, \ldots N-p-1. \end{cases}$$

The corresponding variance is such that $E[(\hat{\varepsilon} - \varepsilon)^2] = N^2/4\pi^2 p^2 SNR(3N-5p)$.

A. Averaging

Since the acquisition range of the proposed estimator is $N/2p$, then it is desirable to select small values of p in order to have the widest range. Therefore, we propose here a method to improve the accuracy for values of $p < N/2$. Indeed, from (13), we have the following recursion formula:

$$y(n+mp) \simeq e^{j2\pi\varepsilon mp/N}e^{-j2\pi mn/N}e^{-j\pi m(m-1)p/N}y(n) \tag{33}$$

where m is an integer such that $mp < N-1$. This equation suggests forming the following estimator:

$$\hat{\varepsilon}_m = \frac{N}{2\pi mp} \angle\left[\left\{\sum_{n=0}^{N-mp-1} w_m(n)r(n+mp)r^*(n)e^{j2\pi mn/N}\right\}e^{j\pi m(m-1)p/N}\right] \tag{34}$$

where

$$\mathbf{w}_m = \frac{\mathbf{i}_{N-mp}\boldsymbol{\Sigma}_m^{-1}}{\mathbf{i}_{N-mp}\boldsymbol{\Sigma}_m^{-1}\mathbf{i}_{N-mp}^T} \tag{35}$$

and $\boldsymbol{\Sigma}_m$ is an $(N-mp) \times (N-mp)$ matrix with entries equal to $\sigma_m(i,l) = 2\delta_{i,l} - \delta_{i,l+mp} - \delta_{i+mp,l}$. Notice that here again this estimator lends itself well to an efficient implementation via the FFT. The acquisition range of the estimator given by (34) is $N/2mp$ the subcarrier spacing. Therefore, an initial acquisition of the CFO has to be performed by $\hat{\varepsilon}_1$, and then, we compensate this value by multiplying the received signal $r(n)$ by $e^{-j2\pi n\hat{\varepsilon}_1/N}$. Finally, the CFO estimate will be given by averaging in an effective way over all the possible values of m, i.e.,

$$\hat{\varepsilon} = \left(\sum_{m=1}^{M}\frac{1}{\widetilde{var}(\hat{\varepsilon}_m)}\right)^{-1}\left[\sum_{m=1}^{M}\frac{1}{\widetilde{var}(\hat{\varepsilon}_m)}\hat{\varepsilon}_m\right] \tag{36}$$

where $M = \lfloor (N-1)/p \rfloor$ ($\lfloor x \rfloor$ returns the integer part of x), and

$$\widetilde{var}(\hat{\varepsilon}_m) = \frac{N^2}{8\pi^2(mp)^2}\frac{1}{\mathbf{i}_{N-mp}\boldsymbol{\Sigma}_m^{-1}\mathbf{i}_{N-mp}^T} = var(\hat{\varepsilon}_m)\text{SNR}.$$

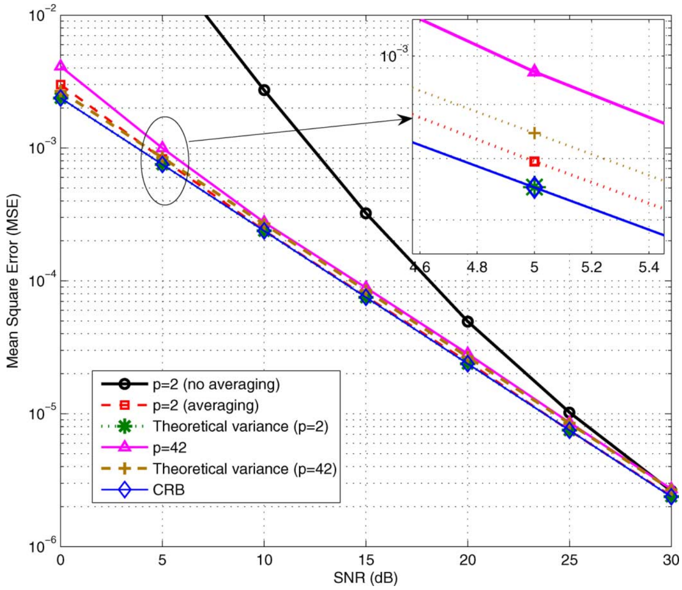

Fig. 1. Mean square error versus SNR over an AWGN channel. [Theoretical variance is given by (29).]

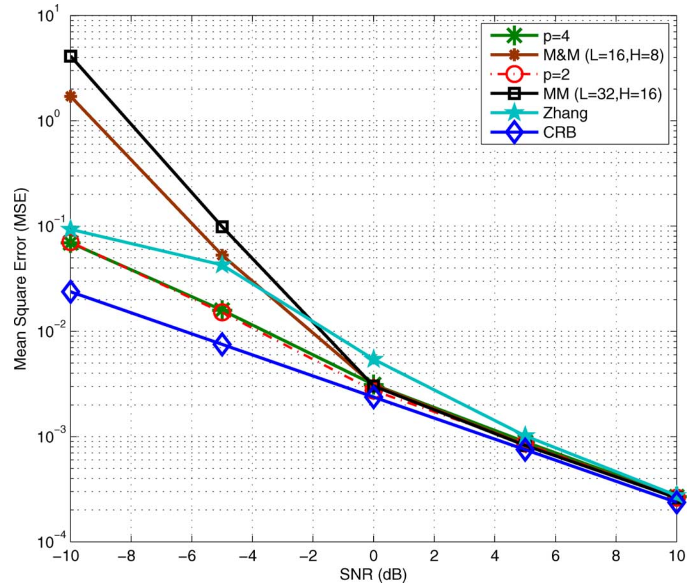

Fig. 2. Mean square error versus SNR over a multipath channel.

VI. Numerical Examples

We consider an OFDM system with 64 subcarriers fed by 16-QAM constellations and a cyclic prefix of 16 samples in length ($N/4$). These are the settings of a typical OFDM-based WLAN. The CFO was fixed to 0.3 (so that it falls within the range of all the estimators, i.e., no acquisition is required), and 2000 independent trials were performed to obtain the mean-square error estimates. Fig. 1 shows the results obtained over an AWGN channel. First, we notice that averaging increases accuracy by as much as 8 dB for an MSE of 8×10^{-3}, with this improvement being much higher at low SNR (results are not shown at very low SNR for the sake of clarity). Another important remark is that the proposed estimators achieve the CRB even for a moderate SNR (the CRB formula can be found in [3]). In particular, the averaged version of the estimator will achieve the CRB at a small SNR (around 5 dB). This figure shows also that if a smaller acquisition range is allowed, choosing $p \geq N/2$ will give accurate results without the use of averaging. In order to show the reliability of our estimation, we made a comparison with the well-known M&M scheme and the recent method developed in [5] (the last method is tuned to its best performance, i.e., $\rho = 1$). These simulations were conducted in a multipath environment having four paths with path delays of 0, 8, 10, and 12 samples (the CP length was set to be superior to the channel length in order to avoid intersymbol interference). The amplitude h_i of the ith path is a Rayleigh random variable and varies independently of the others according to an exponential power delay profile, i.e., $E[|h_i|^2] = exp(-\tau_i)$, where τ_i is the delay of the ith path. The power of each channel realization is then normalized, which corresponds to a perfect power control situation. The obtained results are depicted in Fig. 2 (the CRB in the AWGN channel is also plotted as a benchmark of the performance). When the acquisition range of our method and the M&M one is equal to 16 times the subcarrier spacing, i.e., $p = 2$ for the proposed scheme and $L = 32$ for the M&M method, our estimator clearly outperforms the M&M, especially in the low SNR region—so crucial for multi-antenna transceivers in multi-cellular and multi-access environments—where the gap is remarkable. Indeed, for an MSE equal to 7×10^{-2}, the SNR gain is about 6 dB. For an acquisition range of eight times the subcarrier spacing, i.e., $p = 4$ and $L = 16$, the gap is still huge for low SNR, up to 5 dB in some regions. The superiority of our method compared to the scheme in [5] is also obvious, for instance, for $p = 2$ or $p = 4$ at an MSE equal to 4×10^{-2}, where the SNR gain is around 3 dB.

VII. Conclusion

In this letter, a new CFO estimation scheme for OFDM has been presented. The estimation is based on the transmission of a specially designed synchronization symbol. The particular structure in the received samples of our synchronization symbol allows us to derive a closed-form expression for the CFO. This estimator can provide very high accuracy over a wide acquisition range while keeping a very low computational complexity. Simulations have also proved that this new technique gives higher accuracy than recent methods developed in the literature.

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