Closed-Form Expressions for the Exact Cramér–Rao Bounds of Timing Recovery Estimators From BPSK, MSK and Square-QAM Transmissions

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***Abstract*—In this paper, we derive for the first time analytical expressions for the exact Cramér–Rao lower bounds (CRLB) for symbol timing recovery of binary phase shift keying (BPSK), minimum shift keying (MSK), and square QAM-modulated signals. It is assumed that the transmitted data are completely unknown at the receiver and that the shaping pulse verifies the first Nyquist criterion. Moreover the carrier phase and frequency are considered as unknown nuisance parameters. The time delay remains constant over the observation interval and the received signal is corrupted by additive white Gaussian noise (AWGN). Our new expressions prove that the achievable performance holds irrespective of the true time delay value. Moreover, they corroborate previous attempts to empirically compute the considered bounds thereby enabling their immediate evaluation.**

***Index Terms*—Non-data-aided (NDA) estimation, QAM signals, stochastic Cramér–Rao lower bound (CRLB), symbol timing recovery.**

I. Introduction

In modern communication systems, the received signal is usually sampled once per-symbol interval to recover the transmitted information. But the unknown time delay, introduced by the channel, must be estimated *a priori* in order to sample the signal at the accurate sampling times. In this context, many time delay estimators have been developed to meet this requirement. These estimators can be mainly categorized into two major categories: data-aided (DA) and non-data-aided (NDA) estimators. In DA estimation, *a priori* known symbols are transmitted to assist the estimation process, although the transmission of a known sequence has the drawback of limiting the whole throughput of the system. Whereas, in the NDA mode, the required parameter is blindly estimated assuming the transmitted symbols to be completely unknown. In both cases, the performance of an estimator affects the performance of the entire system. In the case of an unbiased estimation, the variance of the timing error is usually used to evaluate the estimation accuracy. The CRLB is a lower bound on the variance of any unbiased estimator and is often used as a benchmark for the performance evaluation of actual estimators [1], [2]. The computation of this bound has been previously tackled by many authors, under different simplifying assumptions. For instance, assuming the transmitted data to be perfectly known and one can derive the DA CRLB. The modified CRLB (MCRLB), which is also easy to derive, has been introduced in [3] and [4], but unfortunately it departs dramatically from the exact (stochastic) CRLB, especially at low signal-to-noise ratios (SNR).

Actually, the time delay stochastic CRLBs of higher-order modulations were empirically computed in previous works. Their analytical expressions were tackled only for specific SNR regions, i.e., very low or very high-SNR values and the derived bounds are referred to as ACRLBs (asymptotic CRLBs). In fact, in [5], the stochastic CRLB was tackled under the low-SNR assumption and an analytical expression of the considered bound (ACRLB) was derived for arbitrary PSK, QAM, and PAM constellations. In this SNR region, the authors of [5] approximated the likelihood function by a truncated Taylor series expansion to obtain a relatively simple ACRLB expression. An analytical expression was also introduced in [6] under the high-SNR assumption. This high-SNR ACRLB coincides with the stochastic CRLB in this SNR region but unfortunately it cannot be used even for moderate (practical) SNR values. Another approach was later proposed in [7] and [8] to compute the NDA deterministic (or conditional) CRLBs, in which the symbols are considered as deterministic unknown parameters. Then the conditional CRLB is derived from the compressed likelihood function $f(\boldsymbol{y};\boldsymbol{\theta},\hat{\boldsymbol{x}})$ in which $\boldsymbol{y}$ stands for the observed vector, $\boldsymbol{\theta}$ is the parameter vector of interest (including the unknown time delay) and $\hat{\boldsymbol{x}}$ is the maximum likelihood estimate of the transmitted symbols $\boldsymbol{x}$.

However, it is widely known that the conditional CRLB does not provide the actual performance limit (unconditional or stochastic CRLBs). In an other work, the stochastic CRLB was empirically computed [9] assuming perfect phase and frequency synchronization and a time-limited shaping pulse. Later in [10], its computation was tackled in the presence of unknown carrier phase and frequency and pulses that are unlimited in time. Both [9] and [10] simplified the expression of the bounds but ultimately resorted to empirical methods to evaluate the exact CRLB, without providing any closed-form expressions.

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Motivated by these facts, in this work, we derive for the first time analytical expressions for the stochastic CRLBs of symbol timing recovery from BPSK, MSK and square QAM-modulated signals. We consider the general scenario as in [10] in which the carrier phase and frequency offsets are completely unknown at the receiver, and we show that this assumption does not actually affect the performance of a time delay estimator from perfectly frequency- and phase-synchronized received samples. The derivations assume an AWGN-corrupted received signal and a shaping pulse that verifies the first Nyquist criterion. The last assumption is verified in practice for most of the shaping pulses.

This paper is organized as follows. In Section II, we introduce the system model that will be used throughout this article. In Section III, we derive the analytical expression of the stochastic CRLB for any square QAM modulation. Then, in Section IV, we outline the derivation steps of the CRLB in the cases of BPSK and MSK transmissions. Some graphical representations are presented in Section V and, finally, some concluding remarks are drawn out in Section VI.

II. System Model

Consider a traditional communication system where the channel delays the transmitted signal and a zero-mean *proper*[1] AWGN, with an overall power σ^2, corrupts the received signal. In the case of imperfect frequency and phase synchronization, the received signal is expressed as

$$y(t) = \sqrt{E_s}x(t-\tau)e^{j(2\pi f_c t+\theta)} + w(t) \tag{1}$$

where τ is the time delay, θ is the channel distortion phase, f_c is the carrier frequency offset and j is the complex number verifying $j^2 = -1$. The parameters τ, θ and f_c are assumed to be deterministic but unknown. They can be gathered in the following unknown parameter vector:

$$\boldsymbol{\nu} = [\tau, \theta, f_c]^T. \tag{2}$$

In (1), $w(t)$ is a *proper* complex Gaussian white noise with independent real and imaginary parts, each of variance $\sigma^2/2$, and $x(t)$ is the transmitted signal given by

$$x(t) = \sum_{i=1}^{K} a_i h(t-iT), \tag{3}$$

with $\{a_i\}_{i=1}^K$ being the sequence of K transmitted symbols drawn from a BPSK, an MSK or any square-QAM constellation and T is the symbol duration. The transmitted symbols are assumed to be statistically independent and equally likely, with normalized energy, i.e., $E\{|a_i|^2\} = 1$. Finally, $h(t)$ is a square-root Nyquist shaping pulse function with unit-energy which will be seen in Sections III and IV, as would be expected, to have an important impact on the CRLB and therefore on the system's performance. The Nyquist pulse $g(t)$ obtained from $h(t)$ is defined as

$$g(t) = \int_{-\infty}^{+\infty} h(x)h(t+x)dx \tag{4}$$

and satisfies the first Nyquist criterion

$$g(nT) = 0, \quad \text{for any integer } n \neq 0. \tag{5}$$

Suppose that we are able to produce unbiased estimates, $\hat{\boldsymbol{\nu}}$, of the vector $\boldsymbol{\nu}$ from the received signal. Then the CRLB, which verifies $\mathrm{E}\{(\hat{\boldsymbol{\nu}}-\boldsymbol{\nu})^2\} \geq \mathrm{CRLB}\ (\boldsymbol{\nu})$, is defined as [1], [2]

$$\mathrm{CRLB}\ (\boldsymbol{\nu}) = \boldsymbol{I}^{-1}(\boldsymbol{\nu}) \tag{6}$$

where $\boldsymbol{I}(\boldsymbol{\nu})$ is the Fisher information matrix (FIM) whose entries are defined as

$$[\boldsymbol{I}(\boldsymbol{\nu})]_{i;j} = E\left\{\frac{\partial L(\boldsymbol{\nu})}{\partial \nu_i}\frac{\partial L(\boldsymbol{\nu})}{\partial \nu_j}\right\}, \quad i,j = 1,2,3, \tag{7}$$

with $L(\boldsymbol{\nu})$ being the log-likelihood function of the parameters to be estimated and $\{\nu_i\}_{i=1}^3$ are the elements of the unknown parameter vector $\boldsymbol{\nu}$.

To begin with, we show in Appendix A that the problem of time delay estimation is disjoint from the problem of carrier phase and frequency estimation. Indeed, we show that the FIM is block-diagonal structured as follows:

$$\boldsymbol{I}(\boldsymbol{\nu}) = \begin{pmatrix} \mathrm{CRLB}^{-1}(\tau) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_2(\theta, f_c) \end{pmatrix} \tag{8}$$

where $\mathbf{0} = [0,0]^T$, $\mathrm{CRLB}(\tau) = [\boldsymbol{I}(\boldsymbol{\nu})]_{1;1}^{-1}$ is the CRLB of the time delay parameter and $\boldsymbol{I}_2(\theta, f_c)$ is the (2×2) FIM pertaining to the joint estimation of f_c and θ. Hence, we prove analytically that we deal with two separable estimation problems; on one hand, time delay estimation and, on the other hand, carrier phase and frequency estimation. Actually, this conclusion has been already made in [10] but the authors resorted to empirical evaluations to find that the elements $[\boldsymbol{I}(\boldsymbol{\nu})]_{1;2}$ and $[\boldsymbol{I}(\boldsymbol{\nu})]_{1;3}$ of the FIM are almost equal to zero. Now, since the parameters are decoupled, we only need to derive the first element of the global FIM, $[\boldsymbol{I}(\boldsymbol{\nu})]_{1;1}$ in order to find the CRLB for time delay estimation under imperfect frequency and phase synchronization. Therefore, in the following, we consider the virtually derotated received signal $\tilde{y}(t)$ given by

$$\tilde{y}(t) = y(t)e^{-j(2\pi f_c t+\theta)} = \sqrt{E_s}x(t-\tau) + \tilde{w}(t) \tag{9}$$

where $\tilde{w}(t) = w(t)e^{-j(2\pi f_c t+\theta)}$ is also a *proper* AWGN with an overall power σ^2 since the nuisance parameters are assumed to be deterministic.

We mention that $|\cdot|$, $\Re\{\cdot\}$, $\Im\{\cdot\}$ and $\{\cdot\}^*$ return the magnitude, real, imaginary and conjugate of any complex number and $\mathrm{E}\{\cdot\}$ is the statistical expectation. We also define the SNR of the system as $\rho = E_s/\sigma^2$.

[1]A *proper* complex random process $v(t)$ satisfies $E\{v(t)^2\} = 0$.

III. Time Delay CRLB for Square QAM-Modulated Signals

In this section, we introduce the main contribution embodied by this paper which consists in deriving closed-form expressions for the stochastic CRLBs of time delay estimation when the transmitted data are unknown and drawn from any M-ary square QAM-constellation (i.e., $M = 2^{2p}$).

Before further development, it is important to emphasize that an exact representation of $\tilde{y}(t)$ requires an infinite-dimensional vector representation $\tilde{\boldsymbol{y}}$. But let us consider the N-dimensional truncated vectors $\tilde{\boldsymbol{y}}_N$, $\boldsymbol{x}_N$ and $\tilde{\boldsymbol{w}}_N$, representing the projection, over an orthonormal basis of N dimensions, of $\tilde{y}(t)$, $x(t)$ and $\tilde{w}(t)$, respectively. Then, the pdf of $\tilde{\boldsymbol{y}}_N$ conditioned on the transmitted symbols $\boldsymbol{a}$ and parameterized by τ is given by [4]

$$P(\tilde{\boldsymbol{y}}_N \,|\, \boldsymbol{a}; \tau) = \prod_{i=1}^{N} \frac{1}{\pi\sigma^2} \exp\left\{-\frac{|\tilde{y}_k - x_k|^2}{\sigma^2}\right\}. \tag{10}$$

To derive the likelihood function which incorporates all the information contained in $\tilde{y}(t)$, we should make N tend to infinity to get $P(\tilde{\boldsymbol{y}} \,|\, \boldsymbol{a}; \tau)$. However, convergence problems appear. To overcome these problems, $P(\tilde{\boldsymbol{y}}_N \,|\, \boldsymbol{a}; \tau)$ is divided by $1/(\pi\sigma^2)^N \exp\{1/\sigma^2 \sum_{k=1}^{N} |\tilde{y}_k|^2\}$ to obtain

$$\Lambda(\tilde{\boldsymbol{y}} \,|\, \boldsymbol{a}; \tau) = \exp\left\{\frac{2\sqrt{E_s}}{\sigma^2}\sum_{k=1}^{N}\Re\{\tilde{y}_k x_k^*\} - \frac{E_s}{\sigma^2}\sum_{k=1}^{N}|x_k|^2\right\}, \tag{11}$$

and as N tends to infinity, we obtain the conditional likelihood function

$$\Lambda(\tilde{\boldsymbol{y}}|\boldsymbol{a}; \tau) = \exp\left\{\frac{2\sqrt{E_s}}{\sigma^2}\int_{-\infty}^{+\infty}\Re\{\tilde{y}(t)x(t)^* dt\} - \frac{E_s}{\sigma^2}\int_{-\infty}^{+\infty}|x(t)|^2 dt\right\}. \tag{12}$$

To begin with, we note that since the transmitted symbols $\{a_i\}_{i=1}^{K}$ are equally likely, then the desired likelihood function of the derotated observation vector $\tilde{\boldsymbol{y}}$ can be written as

$$\Lambda(\tilde{\boldsymbol{y}}; \tau) = E\left\{\prod_{i=1}^{K} F(a_i, \tilde{y}(t))\right\} \tag{13}$$

where the expectation is performed with respect to the vector of transmitted symbols and

$$F(a_i, \tilde{y}(t)) = \exp\left\{\frac{2\sqrt{E_s}}{\sigma^2}\int_{-\infty}^{+\infty}\Re\{\tilde{y}(t)a_i^*\} \times h(t - iT - \tau)dt - \frac{E_s}{\sigma^2}|a_i|^2\right\}. \tag{14}$$

It can be shown that (13) reduces simply to

$$\Lambda(\tilde{\boldsymbol{y}}; \tau) = \frac{1}{M^K}\prod_{i=1}^{K} H_i(\tau) \tag{15}$$

where

$$H_i(\tau) = \sum_{c_k \in C} \exp\left\{-\frac{E_s}{\sigma^2}|c_k|^2 + \frac{2\sqrt{E_s}}{\sigma^2}\int_{-\infty}^{+\infty}\Re\{\tilde{y}(t)c_k^*\}h(t - iT - \tau)dt\right\} \tag{16}$$

in which C is the constellation alphabet. Actually, the main difficulty in deriving an analytical expression for the stochastic CRLB stems from the complexity of the log-likelihood function. Therefore, we will manipulate the summation involved in (16). In fact, considering only square QAM-modulated signals, we are able, by exploiting the full symmetry of the constellation, to factorize $H_i(\tau)$ which in turn linearizes the global log-likelihood function and ultimately linearizes all the derivations.

Indeed, denoting by $\tilde{C}$ the subset of the alphabet points with positive real and imaginary parts (i.e., $\tilde{C} = \{(2i-1)d_p + j(2k-1)d_p\}_{i,k=1,2,\ldots,2^{p-1}}$), the constellation alphabet C is decomposed as follows:

$$C = \tilde{C} \cup \tilde{C}^* \cup (-\tilde{C}) \cup (-\tilde{C}^*). \tag{17}$$

Note that d_p is the inter-symbol distance derived under the assumption of a normalized-energy square QAM constellation as follows:

$$d_p = \frac{2^{p-1}}{\sqrt{2^p \sum_{k=1}^{2^{p-1}}(2k-1)^2}}. \tag{18}$$

Using (17), we rewrite (16) as

$$\begin{aligned} H_i(\tau) &= \sum_{\tilde{c}_k \in \tilde{C}} \exp\left\{-\frac{E_s}{\sigma^2}|\tilde{c}_k|^2\right\} \\ &\quad \times \left(\exp\left\{\frac{2\sqrt{E_s}}{\sigma^2}\int_{-\infty}^{+\infty}\Re\{\tilde{y}(t)(-\tilde{c}_k^*)\}\ h(t-iT-\tau)dt\right\}\right. \\ &\quad + \exp\left\{\frac{2\sqrt{E_s}}{\sigma^2}\int_{-\infty}^{+\infty}\Re\{\tilde{y}(t)(-\tilde{c}_k)\}\ h(t-iT-\tau)dt\right\} \\ &\quad + \exp\left\{\frac{2\sqrt{E_s}}{\sigma^2}\int_{-\infty}^{+\infty}\Re\{\tilde{y}(t)\tilde{c}_k^*\}\ h(t-iT-\tau)dt\right\} \\ &\quad \left. + \exp\left\{\frac{2\sqrt{E_s}}{\sigma^2}\int_{-\infty}^{+\infty}\Re\{\tilde{y}(t)\tilde{c}_k\}\ h(t-iT-\tau)dt\right\}\right). \end{aligned} \tag{19}$$

Now using the hyperbolic cosine function defined by $2\cosh(x) = e^x + e^{-x}$, (19) reduces simply to

$$\begin{aligned} H_i(\tau) &= 2\sum_{\tilde{c}_k \in \tilde{C}} \exp\left\{-\frac{E_s}{\sigma^2}|\tilde{c}_k|^2\right\} \\ &\quad \times \left[\cosh\left(\frac{2\sqrt{E_s}}{\sigma^2}\int_{-\infty}^{+\infty}\Re\{\tilde{y}(t)\tilde{c}_k^*\}\quad h(t-iT-\tau)dt\right)\right. \\ &\quad \left. + \cosh\left(\frac{2\sqrt{E_s}}{\sigma^2}\int_{-\infty}^{+\infty}\Re\{\tilde{y}(t)\tilde{c}_k\}\quad h(t-iT-\tau)dt\right)\right]. \end{aligned} \tag{20}$$

Moreover, using the fact that $\cosh(a) + \cosh(b) = 2\cosh(\frac{a+b}{2})\cosh(\frac{a-b}{2})$ and noting that $\tilde{c}_k + \tilde{c}_k^* = 2\Re\{\tilde{c}_k\}$ and $\tilde{c}_k - \tilde{c}_k^* = 2j\Im\{\tilde{c}_k\}$, we obtain

$$H_i(\tau) = 2\sum_{\tilde{c}_k\in\tilde{C}} \exp\left\{-\frac{E_s}{\sigma^2}|\tilde{c}_k|^2\right\} \times\cosh\left(\frac{2\sqrt{E_s}}{\sigma^2}\Re\{\tilde{c}_k\}\int_{-\infty}^{+\infty}\Re\{\tilde{y}(t)\}\ h(t-iT-\tau)dt\right) \times\cosh\left(\frac{2\sqrt{E_s}}{\sigma^2}\Im\{\tilde{c}_k\}\int_{-\infty}^{+\infty}\Im\{\tilde{y}(t)\}\ h(t-iT-\tau)dt\right). \tag{21}$$

Recall that $\tilde{C} = \{(2l-1)d_p + j(2m-1)d_p\}_{l,m=1,2,\ldots,2^{p-1}}$ and hence the previous expression of $H_i(\tau)$ is rewritten as

$$\begin{aligned} H_i(\tau) &= 4\sum_{l=1}^{2^{p-1}}\sum_{m=1}^{2^{p-1}}\exp\left\{-\frac{E_s((2l-1)^2+(2m-1)^2)d_p^2}{\sigma^2}\right\} \\ &\times\cosh\left(\frac{2\sqrt{E_s}}{\sigma^2}(2l-1)d_p\int_{-\infty}^{+\infty}\Re\{\tilde{y}(t)\} \times h(t-iT-\tau)dt\right) \\ &\times\cosh\left(\frac{2\sqrt{E_s}}{\sigma^2}(2m-1)d_p\int_{-\infty}^{+\infty}\Im\{\tilde{y}(t)\} \times h(t-iT-\tau)dt\right). \end{aligned} \tag{22}$$

Then, splitting the two sums in (22), $H_i(\tau)$ is factorized as follows[2]:

$$H_i(\tau) = 4F(U_i(\tau))F(V_i(\tau)) \tag{23}$$

where

$$U_i(\tau) = \int_{-\infty}^{+\infty}\Re\{\tilde{y}(t)\}h(t-iT-\tau)dt \tag{24}$$

$$V_i(\tau) = \int_{-\infty}^{+\infty}\Im\{\tilde{y}(t)\}h(t-iT-\tau)dt \tag{25}$$

and

$$F(x) = \sum_{k=1}^{2^{p-1}}\exp\left\{-\frac{E_s}{\sigma^2}(2k-1)^2d_p^2\right\} \times\cosh\left(\frac{2\sqrt{E_s}}{\sigma^2}(2k-1)d_p x\right). \tag{26}$$

Now, injecting the expression of $H_i(\tau)$ in the likelihood function of the received signal (15), we obtain

$$\Lambda(\tilde{\boldsymbol{y}};\tau) = \left(\frac{4}{M}\right)^K\prod_{i=1}^{K}F(U_i(\tau))F(V_i(\tau)). \tag{27}$$

[2]Note that similar factorization was recently used to derive an analytical expression for the NDA SNR estimation [11], [12].

Finally, the log-likelihood function of the received signal expands to

$$L(\tau) = \sum_{i=1}^{K}\ln(F(U_i(\tau))) + \sum_{i=1}^{K}\ln(F(V_i(\tau))). \tag{28}$$

Note from (28) that due to the factorization of $H_i(\tau)$ in (19), the global log-likelihood function of interest in (28) involves the sum of two analogous terms. This reduces considerably the complexity of the stochastic CRLB derivation. In fact, the first derivative of (28) with respect to τ is obtained as follows:

$$\frac{\partial L(\tau)}{\partial\tau} = \sum_{i=1}^{K}\frac{\dot{F}(U_i(\tau))}{F(U_i(\tau))}\frac{\partial U_i(\tau)}{\partial\tau} + \frac{\dot{F}(V_i(\tau))}{F(V_i(\tau))}\frac{\partial V_i(\tau)}{\partial\tau} \tag{29}$$

where $\dot{F}(x) = \frac{\partial F(x)}{\partial x}$ is given by

$$\dot{F}(x) = \sum_{k=1}^{2^{p-1}}\exp\left\{-\rho(2k-1)^2d_p^2\right\}\frac{2\sqrt{\rho}}{\sigma}(2k-1)d_p \times\sinh\left(\frac{2\sqrt{\rho}}{\sigma}(2k-1)d_p x\right). \tag{30}$$

Then, the first diagonal element of the FIM matrix is expressed as

$$\begin{aligned} [\boldsymbol{I}(\boldsymbol{\nu})]_{1;1} &= \mathrm{E}\left\{\left(\frac{\partial L(\tau)}{\partial\tau}\right)^2\right\} \\ &= \mathrm{E}\left\{\sum_{i=1}^{K}\sum_{l=1}^{K}\frac{\dot{F}(U_i(\tau))}{F(U_i(\tau))}\frac{\dot{F}(U_l(\tau))}{F(U_l(\tau))}\dot{U}_i(\tau)\dot{U}_l(\tau)\right\} \\ &+ 2\mathrm{E}\left\{\sum_{i=1}^{K}\sum_{l=1}^{K}\frac{\dot{F}(U_i(\tau))}{F(U_i(\tau))}\frac{\dot{F}(V_l(\tau))}{F(V_l(\tau))}\dot{U}_i(\tau)\dot{V}_l(\tau)\right\} \\ &+ \mathrm{E}\left\{\sum_{i=1}^{K}\sum_{l=1}^{K}\frac{\dot{F}(V_i(\tau))}{F(V_i(\tau))}\frac{\dot{F}(V_l(\tau))}{F(V_l(\tau))}\dot{V}_i(\tau)\dot{V}_l(\tau)\right\} \end{aligned} \tag{31}$$

where $\dot{U}_l(\tau)$ and $\dot{V}_l(\tau)$ are the derivatives of $U_l(\tau)$ and $V_l(\tau)$ with respect to τ.

Starting from (31), the derivation of $[\boldsymbol{I}(\boldsymbol{\nu})]_{1;1}$ involves the evaluation of three expectations. However, it is easy to verify that the first and the last expectations in the right-hand side of (31) are performed with respect to two random processes having the same statistical properties and they are therefore identically equal. Moreover, as shown in Appendix B, the second expectation is equal to zero. Therefore, (31) reduces simply to

$$[\boldsymbol{I}(\boldsymbol{\nu})]_{1;1} = 2\sum_{i=1}^{K}\sum_{l=1}^{K}E\left\{\frac{\dot{F}(U_i(\tau))}{F(U_i(\tau))}\frac{\dot{F}(U_l(\tau))}{F(U_l(\tau))}\dot{U}_i(\tau)\dot{U}_l(\tau)\right\}. \tag{32}$$

First, we consider the case where $i = l$, and we show in Appendix C that $U_i(\tau)$ and $\dot{U}_i(\tau)$ are statistically independent. This results in

$$E\left\{\left(\frac{\dot{F}(U_i(\tau))}{F(U_i(\tau))}\right)^2 (\dot{U}_i(\tau))^2\right\} = E\left\{\left(\frac{\dot{F}(U_i(\tau))}{F(U_i(\tau))}\right)^2\right\} E\{(\dot{U}_i(\tau))^2\}. \quad (33)$$

These two expectations involved in the right-hand side of (33) are easily evaluated as follows:

$$\mathrm{E}\left\{\left(\frac{\dot{F}(U_i(\tau))}{F(U_i(\tau))}\right)^2\right\} = \int_{-\infty}^{+\infty} \left(\frac{\dot{F}(U_i(\tau))}{F(U_i(\tau))}\right)^2 P(U_i(\tau))dU_i(\tau) = \frac{1}{\sqrt{\pi\sigma^2}}\sqrt{\frac{4}{M}} \int_{-\infty}^{+\infty} \frac{\dot{F}^2(U_i(\tau))}{F(U_i(\tau))} \times \exp\left\{-\frac{U_i^2(\tau)}{\sigma^2}\right\} dU_i(\tau) \quad (34)$$

$$E\{(\dot{U}_i(\tau))^2\} = \frac{E_s}{2}\sum_{k=1}^{K} \dot{g}^2((i-k)T) - \frac{\sigma^2}{2}\ddot{g}(0) \quad (35)$$

where $\dot{g}(\cdot)$ and $\ddot{g}(\cdot)$ are the first and second derivative of $g(\cdot)$, respectively. We simplify (34) by changing $\sqrt{2}U_i(\tau)/\sigma$ by x and we obtain the following result:

$$\mathrm{E}\left\{\left(\frac{\dot{F}(U_i(\tau))}{F(U_i(\tau))}\right)^2\right\} = \sqrt{\frac{2}{\pi M}}\frac{\rho}{\sigma^2} \int_{-\infty}^{+\infty} \frac{g_\rho^2(x)}{G_\rho(x)} e^{-\frac{x^2}{2}} dx \quad (36)$$

where

$$g_\rho(x) = \sum_{k=1}^{2^{p-1}} \exp\left\{-\rho(2k-1)^2 d_p^2\right\} \sqrt{2}(2k-1)d_p \times \sinh(\sqrt{2\rho}(2k-1)d_p x) \quad (37)$$

$$G_\rho(x) = \sum_{k=1}^{2^{p-1}} \exp\left\{-\rho(2k-1)^2 d_p^2\right\} \times \cosh(\sqrt{2\rho}(2k-1)d_p x). \quad (38)$$

We now consider the case where $i \neq l$. The intersymbol interference results in a statistical dependence between $U_i(\tau)$ and the first derivatives $\dot{U}_i(\tau)$ and $\dot{U}_l(\tau)$ (likewise for $U_l(\tau)$ and the first derivatives $\dot{U}_i(\tau)$ and $\dot{U}_l(\tau)$). Thus, using a standard probobility approach to derive the expectations involved in (32), we first average by conditioning on $U_i(\tau)$ and $U_l(\tau)$, then average the resulting expression with respect to these two random variables. To that end, consider the expectation of $\dot{U}_i(\tau)$ and $\dot{U}_l(\tau)$ conditioned on $U_i(\tau)$ and $U_l(\tau)$:

$$E\{\dot{U}_i(\tau)\,|\,U_i(\tau), U_l(\tau)\} = U_l(\tau)\dot{g}((i-l)T) \quad (39)$$
$$E\{\dot{U}_l(\tau)\,|\,U_i(\tau), U_l(\tau)\} = U_i(\tau)\dot{g}((l-i)T). \quad (40)$$

Using (39) and (40), it follows that

$$E\left\{\frac{\dot{F}(U_i(\tau))}{F(U_i(\tau))}\frac{\dot{F}(U_l(\tau))}{F(U_l(\tau))}\dot{U}_i(\tau)\dot{U}_l(\tau)\,\middle|\, U_i(\tau), U_l(\tau)\right\} = \frac{\dot{F}(U_i(\tau))}{F(U_i(\tau))} \times \frac{\dot{F}(U_l(\tau))}{F(U_l(\tau))}U_i(\tau)U_l(\tau)\dot{g}((i-l)T)\dot{g}((l-i)T) \quad (41)$$

and we obtain

$$E\left\{\frac{\dot{F}(U_i(\tau)}{F(U_i(\tau))}\frac{\dot{F}(U_l(\tau))}{F(U_l(\tau))}\dot{U}_i(\tau)\dot{U}_l(\tau)\right\} = -\left(E\left\{\frac{\dot{F}(U_i(\tau))}{F(U_i(\tau))}U_i(\tau)\right\}\right)^2 \dot{g}^2((i-l)T) \quad (42)$$

where the last equality follows from the statistical independence of $U_i(\tau)$ and $U_l(\tau)$ and

$$E\left\{\frac{\dot{F}(U_i(\tau))}{F(U_i(\tau))}U_i(\tau)\right\} = \sqrt{\frac{2\rho}{\pi M}} \int_{-\infty}^{+\infty} x g_\rho(x) e^{-\frac{x^2}{2}} dx. \quad (43)$$

Finally, gathering all these results, we obtain the analytical expression of the stochastic CRLB for symbol timing estimation. From square QAM-modulated signals in the presence of carrier phase and frequency offsets as follows:

$$\mathrm{CRLB}(\tau) = \left[\left(2\rho^2\sum_{m=1}^{K}\sum_{n=1}^{K}\dot{g}^2((m-n)T) - 2K\rho\ddot{g}(0)\right) \times \sqrt{\frac{2}{\pi M}}\int_{-\infty}^{+\infty}\frac{g_\rho^2(x)}{G_\rho(x)}e^{-\frac{x^2}{2}}dx - \frac{4\rho}{\pi M}\left(\int_{-\infty}^{+\infty} x g_\rho(x) e^{-\frac{x^2}{2}} dx\right)^2 \times \sum_{m=1}^{K}\sum_{n=1}^{K}\dot{g}^2((m-n)T)\right]^{-1}. \quad (44)$$

Note that for large values of K, one can use the following accurate approximation [10]:

$$\sum_{m=1}^{K}\sum_{n=1}^{K}\dot{g}^2((m-n)T) \approx K\sum_{m=-\infty}^{+\infty}\dot{g}^2(mT). \quad (45)$$

It is worth mentioning that the new analytical expression in (44) allows the immediate evaluation of time delay stochastic CRLBs, contrarily to the empirical approaches presented in [9] and [10], and this is made possible for any square QAM modulation order. Second, the shaping pulse is involved only via $\ddot{g}(0)$ and $\dot{g}^2((m-n)T)$, and is separate from the factors resulting from the modulation order. Moreover, to the best of our knowledge, we show here for the first time, through our new analytical expression, that the true value of the time delay parameter does not affect the actual achievable performance as intuitively expected, i.e., the variance of the estimation error holds irrespective of the time delay value to be estimated.

IV. CRLB for BPSK and MSK Modulated Signals

In this section, we consider the BPSK and MSK modulations. In BPSK transmissions, the data symbols take values in $\{-1,+1\}$ with equal probabilities. In MSK transmissions, the symbols are defined as $a_{k+1}=ja_kc_k$ where c_k is a sequence of BPSK symbols and a_0 is the original value drawn from the set $\{-1,-j,+1,+j\}$. For these two transmission schemes, the key derivation steps of the NDA CRLB will be briefly outlined in the following. All derivation details can be found in Appendix D.

First, the likelihood function of interest based on the received signal is:

$$\Lambda(\tilde{\boldsymbol{y}};\tau)=\prod_{i=1}^{K}\cosh\left(\frac{2\sqrt{E_s}}{\sigma^2}\times\int_{-\infty}^{+\infty}\Re\{b_i^*\tilde{y}(t)\}h(t-iT-\tau)dt+\frac{E_s}{\sigma^2}|b_i|^2\right) \quad (46)$$

where b_i is equal to 1 and $j^{i-1}a_0$ for BPSK and MSK, respectively. Therefore, we show that the useful log-likelihood function of $\tilde{\boldsymbol{y}}$ is given by

$$L(\tau)=\sum_{i=1}^{K}\ln\left(\cosh\left(\frac{2\sqrt{E_s}}{\sigma^2}\int_{-\infty}^{+\infty}\Re\{b_i^*\tilde{y}(t)\}\times h(t-iT-\tau)dt\right)\right). \quad (47)$$

Note that $\tilde{y}(t)$ is defined in (9). After some algebraic manipulations, detailed in Appendix D, it turns out that the analytical expression of the stochastic CRLB for time delay estimation is the same for BPSK and MSK modulations, and it is given by:

$$\mathrm{CRLB}=\frac{1}{4\rho}\left[\left(1-\frac{1}{\sqrt{2\pi}}e^{-\rho}\beta(\rho)\right)\times\left(\rho\sum_{m=1}^{K}\sum_{n=1}^{K}\dot{g}((m-n)T)-\frac{K}{2}\ddot{g}(0)\right)-\rho\sum_{m=1}^{K}\sum_{n=1}^{K}\dot{g}((m-n)T)\right]^{-1} \quad (48)$$

where $\beta(\cdot)$ is defined as

$$\beta(\rho)=\int_{-\infty}^{+\infty}\frac{e^{-\frac{x^2}{2}}}{\cosh(\sqrt{2\rho}x)}dx. \quad (49)$$

V. Graphical Representations

In this section, we provide graphical representations of the time delay CRLBs and the CRLB/MCRLB ratio for different modulation orders. First, we mention that the even integrand functions $\frac{g_\rho^2(x)}{G_\rho(x)}e^{-\frac{x^2}{2}}$, $xg_\rho(x)e^{-\frac{x^2}{2}}$ and $\frac{e^{-\frac{x^2}{2}}}{\cosh(\sqrt{2\rho}x)}$ involved in (44) and (49), respectively, decrease rapidly as $|t|$ increases. Therefore, the integrals over $[-\infty,+\infty]$ can be accurately approximated by a finite integral over an interval $[-A,A]$ and the Riemann integration method can be adequately used. In our simulations, we note that $A=100$ and a summation step of 0.5 provided accurate values for the infinite integral.

First, we plot in Fig. 1 the CRLBs for different modulation orders and compare them to the ones previously obtained empirically in [10]. We see a good agreement between the two approaches thereby validating the developments above. Then, we confirm through Fig. 2 that, at low SNR values, the MCRLB is a looser bound compared to the exact CRLB. Indeed, this figure depicts the CRLB/MCRLB ratio as a function of the SNR. This ratio quantifies the performance degradation that arises from randomizing the transmitted data and it approaches 1 at high SNR values. Hence, in this SNR region, the MCRLB can be used as a benchmark to evaluate the performance of unbiased time delay estimators instead of the exact CRLB, since it is easier to evaluate. However, the gap between the two bounds becomes important as soon as the SNR drops below 7 dB, even for QPSK-modulated signals, where the stochastic CRLB quantifies the actual performance limit. Moreover, we consider in this figure two values of the roll-off factor, 0.2 and 1, in order to illustrate the effect of the roll-off factor on timing estimation. Clearly, timing estimation is less accurate at a lower roll-off factor (larger intersymbol interference). Moreover, we see from Fig. 3 that the different CRLBs tend to ultimately coincide with the MCRLB as long as the SNR gets increases. This actually, in the high SNR region, the achievable performance of NDA estimation of the signal time delay is equivalent to the one obtained when the received symbols are perfectly known since in this SNR range the MCRLB coincides with the DA CRLB. In the specific case where $h(t)$ is time limited to the symbol duration, the corresponding CRLB follows directly from the general expression in (44) by taking $\dot{g}(mT)=0$ for all $m\in\mathbb{Z}$:

$$\mathrm{CRLB}(\tau)=\left[-2K\rho\ddot{g}(0)\sqrt{\frac{2}{\pi M}}\int_{-\infty}^{+\infty}\frac{g_\rho^2(x)}{G_\rho(x)}e^{-\frac{x^2}{2}}dx\right]^{-1}. \quad (50)$$

Note from (50) that the resulting CRLB becomes the product of two separate terms; one depending on the shaping pulse function and the other on the signal modulation. This special bound is plotted in Fig. 4. We see again a good agreement in this special case between the CRLBs obtained from our analytical expression in (50) and their empirical counterparts plotted in Fig. 1 of [9]. This particular expression still finds applications in many conventional systems and in the emerging impulse radio technology [13], [14] where, precisely, synchronization stands today as a very challenging issue.

VI. Conclusion

In this paper, we derived, for the first time, analytical expressions of the Cramér–Rao lower bound for symbol timing estimation in the cases of BPSK, MSK and square-QAM modula-

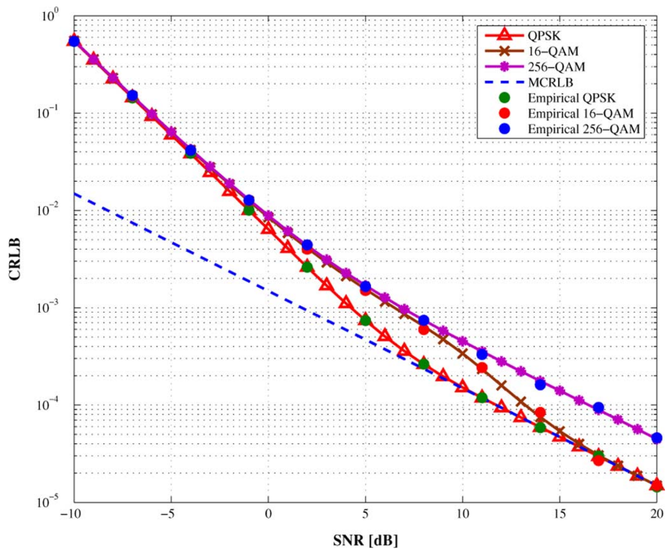

Fig. 1. Compression between the empirical CRLB and the analytical expression in (44) for different modulation orders using $K = 100$ and a raised-cosine pulse with roll-off factor of 0.2.

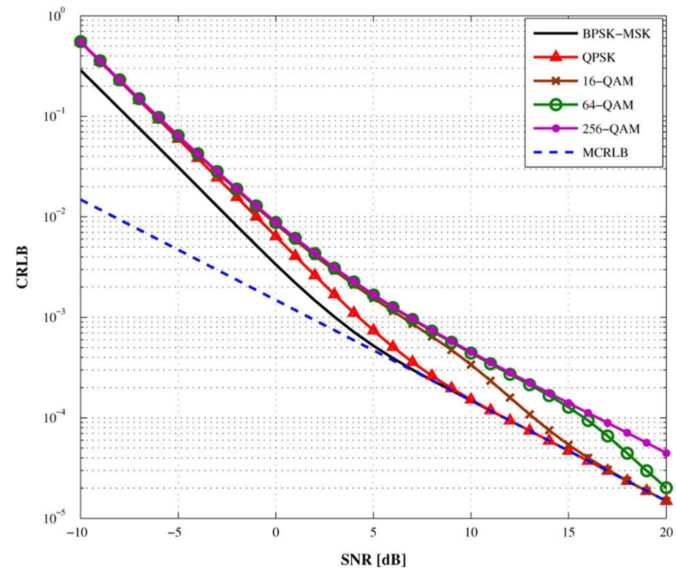

Fig. 3. CRLB versus SNR for different modulation orders using $K = 100$ and a raised-cosine pulse with roll-off factor of 0.2.

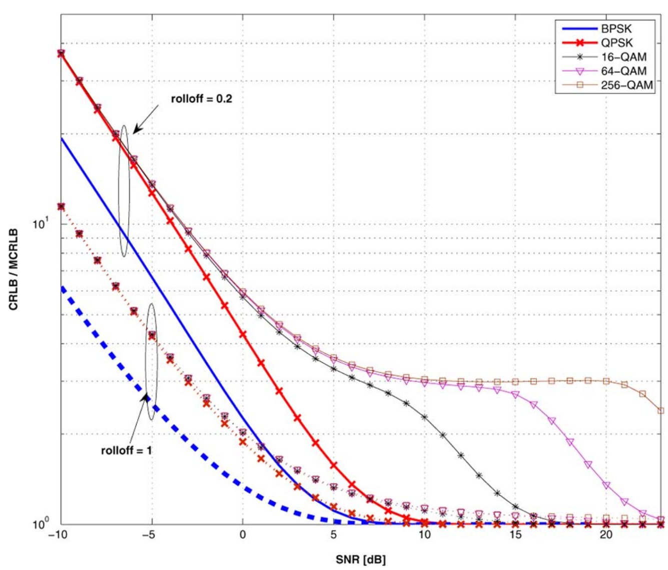

Fig. 2. CRLB/MCRLB ratio versus SNR for different modulation orders using $K = 100$ and a raised-cosine pulse with roll-off factor of 0.2 and 1.

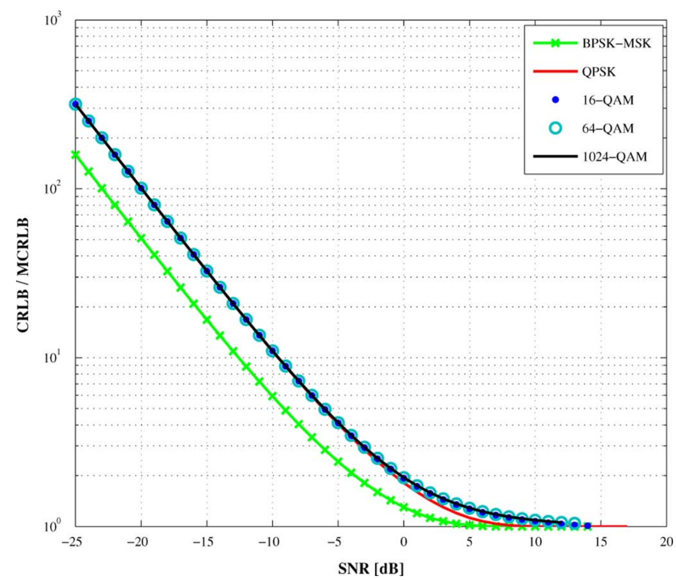

Fig. 4. CRLB/MCRLB ratio versus SNR for different modulations and a time-limited shaping pulse.

tions. We considered the stochastic CRLB where the transmitted data are unknown and randomly drawn. The carrier phase and frequency offsets are also supposed to be unknown (nuisance parameters). We showed that the knowledge of the phase and frequency does not bring any additional information to the time delay estimation problem and that the latter is decoupled from the joint estimation of the carrier frequency and phase offsets. Moreover, our analytical expressions for the CRLBs underline the fact that these bounds do not depend on the time delay value, which used to be stated only intuitively. We confirmed also that the modified CRLB is a valid approximation of the exact CRLB in the high SNR region and that it can be used as a benchmark since it is easier to evaluate. Furthermore, the derived analytical expressions corroborate previous works that empirically computed the stochastic CRLBs via Monte Carlo simulations, and hence provide a useful tool for a quick and easy evaluation of the CRLBs with BPSK, MSK and square-QAM modulations.

Appendix A
Proof of the Block-Diagonal Structure of the FIM

To show that τ and $\boldsymbol{u} = [f_c, \theta]^T$ are decoupled, we consider the actual received signal $y(t)$ instead of the virtually derotated signal $\tilde{y}(t)$. Then we follow the same derivation steps from (13) to (28) to retrieve the log-likelihood function parameterized by $\boldsymbol{\nu}$ as follows:

$$L(\boldsymbol{\nu}) = \sum_{i=1}^{K} \ln(F(U_i(\boldsymbol{\nu}))) + \sum_{i=1}^{K} \ln(F(V_i(\boldsymbol{\nu}))). \tag{51}$$

The first derivatives of this function with respect to the lth element of $\boldsymbol{u}$, $\{u_l\}_{l=1}^{l=2}$, and τ are, respectively, given by

$$\frac{\partial L(\boldsymbol{\nu})}{\partial u_l} = \sum_{i=1}^{K} \frac{\dot{F}(U_i(\boldsymbol{\nu}))}{F(U_i(\boldsymbol{\nu}))} \frac{\partial U_i(\boldsymbol{\nu})}{\partial u_l} + \frac{\dot{F}(V_i(\boldsymbol{\nu}))}{F(V_i(\boldsymbol{\nu}))} \frac{\partial V_i(\boldsymbol{\nu})}{\partial u_l}, \quad l = 1, 2 \tag{52}$$

and

$$\frac{\partial L(\boldsymbol{\nu})}{\partial \tau} = \sum_{i=1}^{K} \frac{\dot{F}(U_i(\boldsymbol{\nu}))}{F(U_i(\boldsymbol{\nu}))} \frac{\partial U_i(\boldsymbol{\nu})}{\partial \tau} + \frac{\dot{F}(V_i(\boldsymbol{\nu}))}{F(V_i(\boldsymbol{\nu}))} \frac{\partial V_i(\boldsymbol{\nu})}{\partial \tau} \tag{53}$$

where $\dot{F}(\,\cdot\,)$ is defined in (30). Then we average $\frac{\partial L(\boldsymbol{\nu})}{\partial \tau}\frac{\partial L(\boldsymbol{\nu})}{\partial \boldsymbol{u}(l)}$ as in (31) to obtain the following result:

$$\begin{aligned}[\boldsymbol{I}(\boldsymbol{\nu})]_{1;l+1} &= \mathrm{E}\left\{\frac{\partial L(\boldsymbol{\nu})}{\partial \tau}\frac{\partial L(\boldsymbol{\nu})}{\partial u_l}\right\} \\ &= 2\sum_{i=1}^{K}\sum_{m=1}^{K} E\left\{\frac{\dot{F}(U_i(\boldsymbol{\nu}))}{F(U_i(\boldsymbol{\nu}))}\frac{\dot{F}(U_m(\boldsymbol{\nu}))}{F(U_m(\boldsymbol{\nu}))}\right. \\ &\qquad \left. \times \frac{\partial U_i(\boldsymbol{\nu})}{\partial \tau}\frac{\partial U_m(\boldsymbol{\nu})}{\partial u_l}\right\}. \end{aligned} \tag{54}$$

In order to simplify the calculations, without loss of generality, we consider $l = 1$. To begin with, we first differentiate $U_m(\boldsymbol{\nu})$ with respect to f_c and we obtain

$$\begin{aligned}\frac{\partial U_m(\boldsymbol{\nu})}{\partial f_c} &= 2\pi \int_{-\infty}^{+\infty} \Im\left\{y(t)e^{-j(2\pi f_c t+\theta)}\right\} \\ &\quad \times h(t - mT - \tau)tdt \\ &= 2\pi\sqrt{E_s}\sum_{n=1}^{K}\Im\{a_m\}\int_{-\infty}^{+\infty} h(t - nT - \tau) \\ &\quad \times h(t - mT - \tau)tdt \\ &\quad + \int_{-\infty}^{+\infty} \Im\{\tilde{w}(t)\}h(t - mT - \tau)tdt. \end{aligned} \tag{55}$$

$\frac{\partial U_m(\boldsymbol{\nu})}{\partial f_c}$ is a function of the imaginary part of the transmitted symbols and the derotated noise, which are mutually independent from the real part of the transmitted symbols and the derotated noise. As a result, $\frac{\partial U_m(\boldsymbol{\nu})}{\partial f_c}$ is independent from $U_i(\boldsymbol{\nu})$, $U_m(\boldsymbol{\nu})$ and $\frac{\partial U_i(\boldsymbol{\nu})}{\partial \tau}$. This allows us to split the expectations in (54):

$$\begin{aligned} &E\left\{\frac{\dot{F}(U_i(\boldsymbol{\nu}))}{F(U_i(\boldsymbol{\nu}))}\frac{\dot{F}(U_m(\boldsymbol{\nu}))}{F(U_m(\boldsymbol{\nu}))}\frac{\partial U_i(\boldsymbol{\nu})}{\partial \tau}\frac{\partial U_m(\boldsymbol{\nu})}{\partial u_l}\right\} \\ &\quad = E\left\{\frac{\dot{F}(U_i(\boldsymbol{\nu}))}{F(U_i(\boldsymbol{\nu}))}\frac{\dot{F}(U_m(\boldsymbol{\nu}))}{F(U_m(\boldsymbol{\nu}))}\frac{\partial U_i(\boldsymbol{\nu})}{\partial \tau}\right\} E\left\{\frac{\partial U_m(\boldsymbol{\nu})}{\partial u_l}\right\}. \end{aligned} \tag{56}$$

Noting that the last expectation is equal to zero, it follows immediately that $[\boldsymbol{I}(\boldsymbol{\nu})]_{1;2}$ is also equal to zero. Thus, we show analytically that the two parameters τ and f_c are decoupled. The same manipulations are used to prove that τ and θ are also decoupled. Therefore, the FIM is block-diagonal structured as given by (8).

APPENDIX B
PROOF OF $E\{\frac{\dot{F}(U_i(\tau))}{F(U_i(\tau))}\frac{\dot{F}(V_l(\tau))}{F(V_l(\tau))}\dot{U}_i(\tau)\dot{V}_l(\tau)\} = 0$

In the following, we briefly show that $E\{\frac{\dot{F}(U_i(\tau))}{F(U_i(\tau))}\frac{\dot{F}(V_l(\tau))}{F(V_l(\tau))}\dot{U}_i(\tau)\dot{V}_l(\tau)\} = 0$. By definition, $U_i(\tau)$ depends on the real part of $\tilde{y}(t)$, while $V_l(\tau)$ involves the imaginary part of $\tilde{y}(t)$, which are statistically independent. It follows that $U_i(\tau)$ and $V_l(\tau)$ are independent. The same arguments hold to show the statistical independence of $\dot{U}_i(\tau)$ and $\dot{V}_l(\tau)$. Then, it immediately follows that

$$\begin{aligned} &E\left\{\frac{\dot{F}(U_i(\tau))}{F(U_i(\tau))}\frac{\dot{F}(V_l(\tau))}{F(V_l(\tau))}\dot{U}_i(\tau)\dot{V}_l(\tau)\right\} \\ &\quad = E\left\{\frac{\dot{F}(U_i(\tau))}{F(U_i(\tau))}\dot{U}_i(\tau)\right\} E\left\{\frac{\dot{F}(V_l(\tau))}{F(V_l(\tau))}\dot{V}_l(\tau)\right\}. \end{aligned} \tag{57}$$

And since $\dot{U}_i(\tau)$ and $U_i(\tau)$ are statistically independent (see Appendix C), each with mean zero, we obtain

$$E\left\{\frac{\dot{F}(U_i(\tau))}{F(U_i(\tau))}\frac{\dot{F}(V_l(\tau))}{F(V_l(\tau))}\dot{U}_i(\tau)\dot{V}_l(\tau)\right\} = 0. \tag{58}$$

APPENDIX C

A. PDFS OF $U_i(\tau)$ AND $V_i(\tau)$

In this Appendix, we establish the joint pdf of $U_i(\tau)$ and $V_i(\tau)$ defined, respectively, in (24) and (25). To that end, we define the *proper* complex random variable $Z_i(\tau) = \int_{-\infty}^{+\infty} \tilde{y}(t)h(t - iT - \tau)dt$. It can be easily seen that $Z_i(\tau) = U_i(\tau) + jV_i(\tau)$ and that $P(Z_i(\tau)) = P(U_i(\tau), V_i(\tau))$. Using the same algebraic manipulations from (16) through (23), we establish the pdf of $Z_i(\tau)$ as follows:

$$\begin{aligned} P(Z_i(\tau)) &= \frac{1}{M}\frac{1}{\pi\sigma^2}\exp\left\{-\frac{|Z_i(\tau)|^2}{\sigma^2}\right\}H_i(\tau) \\ &= \frac{4}{M\pi\sigma^2}\exp\left\{-\frac{U_i^2(\tau) + V_i^2(\tau)}{\sigma^2}\right\} \\ &\quad \times F(U_i(\tau))F(V_i(\tau)) \\ &= P(U_i(\tau))P(V_i(\tau)) \end{aligned} \tag{59}$$

where

$$P(U_i(\tau)) = \frac{1}{\sqrt{\pi\sigma^2}}\sqrt{\frac{4}{M}}\exp\left\{-\frac{U_i^2(\tau)}{\sigma^2}\right\}F(U_i(\tau)) \tag{60}$$

$$P(V_i(\tau)) = \frac{1}{\sqrt{\pi\sigma^2}}\sqrt{\frac{4}{M}}\exp\left\{-\frac{V_i^2(\tau)}{\sigma^2}\right\}F(V_i(\tau)). \tag{61}$$

Note that the factorization of the joint pdf $P(U_i(\tau), V_i(\tau))$ of $U_i(\tau)$ and $V_i(\tau)$ to their elementary pdfs confirms that these are two independent random variables.

B. PROOF OF STATISTICAL INDEPENDENCE OF $U_i(\tau)$ AND $\dot{U}_i(\tau)$

First, note that, $U_i(\tau)$ can be written as

$$U_i(\tau) = \sqrt{E_s}\Re\{a_i\} + \beta_i \tag{62}$$

where

$$\beta_i = \int_{-\infty}^{+\infty} \Re\{\tilde{w}(t)\} h(t - iT - \tau) dt. \tag{63}$$

Therefore, $\dot{U}_i(\tau)$ is given by

$$\begin{aligned}\dot{U}_i(\tau) &= \sqrt{E_s} \sum_{m=1}^{K} \Re\{a_m\} \dot{g}((i-m)T) \\ &\quad - \int_{-\infty}^{+\infty} \Re\{\tilde{w}(t)\} \dot{h}(t - iT - \tau) dt \\ &= \sqrt{E_s} \sum_{m=1}^{K} \Re\{a_m\} \dot{g}((i-m)T) - \dot{\beta}_i. \end{aligned} \tag{64}$$

In addition, $\Re\{a_i\}$ and $\dot{\beta}_i$ are independent since the noise and the transmitted symbols are independent. Recall also that $\dot{g}(0) = 0$ (the maximum of $g(x)$ is located at 0). Then, $\sum_{m=1}^{K} \Re\{a_m\} \dot{g}((i-m)T)$ and $\Re\{a_i\}$ are also independent. Moreover, β_i and $\dot{\beta}_i$ are obtained by a linear transformation of the Gaussian process $\Re\{w(t)\}$. Hence, they are also Gaussian processes. Then, since the cross-correlation of β_i and $\dot{\beta}_i$ is equal to zero, as shown below:

$$\begin{aligned}\mathrm{E}\{\beta_i \dot{\beta}_i\} &= \mathrm{E}\left\{ \iint_{-\infty}^{+\infty} \Re\left\{ w(t_1) e^{-j(2\pi f_c t_1 + \theta)} \right\} \right. \\ &\quad \times \Re\left\{ w(t_2) e^{-j(2\pi f_c t_2 + \theta)} \right\} h(t_1 - iT - \tau) \\ &\quad \left. \times\ \dot{h}(t_2 - iT - \tau) dt_1 dt_2 \right\} \\ &= \frac{\sigma^2}{2} \iint_{-\infty}^{+\infty} \delta(t_1 - t_2) h(t_1) \dot{h}(t_2) dt_1 dt_2 \\ &= \frac{\sigma^2}{2} \dot{g}(0) \\ &= 0, \end{aligned} \tag{65}$$

then, β_i and $\dot{U}_i(\tau)$ are actually two uncorrelated Gaussian random processes and therefore they are independent. Thus, $U_i(\tau)$ and $\dot{U}_i(\tau)$ are independent.

Appendix D
Derivation of the Analytical Expressions for the CRLBs in Case of BPSK and MSK Modulations

Starting from the expression of the log-likelihood function given in (47), we will consider the two cases of BPSK and MSK separately. Starting with BPSK-modulated signals, we show that the log-likelihood function in (47) reduces to

$$L(\tau) = \sum_{i=1}^{K} \ln\left(\cosh\left(\frac{2\sqrt{E_s}}{\sigma^2} U_i(\tau) \right) \right) \tag{66}$$

where

$$U_i(\tau) = \int_{-\infty}^{+\infty} \Re\{\tilde{y}(t)\} h(t - iT - \tau) dt. \tag{67}$$

Then, the first derivative of the log-likelihood function with respect to the time delay parameter, τ, is given by

$$\frac{\partial L(\tau)}{\partial \tau} = \frac{2\sqrt{E_s}}{\sigma^2} \sum_{i=1}^{K} \frac{\sinh\left(\frac{2\sqrt{E_s}}{\sigma^2} U_i(\tau) \right)}{\cosh\left(\frac{2\sqrt{E_s}}{\sigma^2} U_i(\tau) \right)} \dot{U}_i(\tau) \tag{68}$$

where $\dot{U}_i(\tau)$ denotes the first derivative of $U_i(\tau)$ with respect to τ. It is easy to see that

$$\begin{aligned}\dot{U}_i(\tau) &= \sqrt{E_s} \sum_{m=1}^{K} a_m \dot{g}((i-m)T) \\ &\quad - \int_{-\infty}^{+\infty} \Re\{\tilde{w}_i(t)\} \dot{h}(t - iT - \tau) dt. \end{aligned} \tag{69}$$

Now injecting (68) in (7), we obtain

$$\begin{aligned}[\boldsymbol{I}(\boldsymbol{\nu})]_{1;1} &= 4\frac{E_s}{\sigma^4} \sum_{i=1}^{K} \sum_{l=1}^{K} E\left\{ \tanh\left(\frac{2\sqrt{E_s}}{\sigma^2} U_i(\tau) \right) \right. \\ &\quad \left. \times \tanh\left(\frac{2\sqrt{E_s}}{\sigma^2} U_l(\tau) \right) \dot{U}_i(\tau) \dot{U}_l(\tau) \right\} \\ &= 4\frac{E_s}{\sigma^4} \sum_{i=1}^{K} \sum_{l=1}^{K} E_{i,l}. \end{aligned} \tag{70}$$

Note that (70) is similar to (32) (obtained in the case of square QAM modulations). Thus, for the same reasons, it is more convenient to separate the cases when $i = l$ and $i \neq l$. Moreover, it can be shown that the pdf of $U_i(\tau)$ is given by

$$\begin{aligned}P(U_i(\tau)) &= \frac{1}{\sqrt{\pi\sigma^2}} \exp\left\{ -\frac{U_i^2(\tau) + E_s}{\sigma^2} \right\} \\ &\quad \times \cosh\left(\frac{2\sqrt{E_s}}{\sigma^2} U_i(\tau) \right). \end{aligned} \tag{71}$$

Thus, it can be shown that, after some manipulations, the expectations involved in (70) reduce to

$$\begin{aligned}&E\left\{ \tanh^2\left(\frac{2\sqrt{E_s}}{\sigma^2} U_i(\tau) \right) \right\} \\ &\quad = \frac{\exp\left\{ -\frac{E_s}{\sigma^2} \right\}}{\sqrt{\pi\sigma^2}} \\ &\qquad \times \int_{-\infty}^{+\infty} \frac{\sinh^2\left(\frac{2\sqrt{E_s}}{\sigma^2} U \right)}{\cosh\left(\frac{2\sqrt{E_s}}{\sigma^2} U \right)} \exp\left\{ -\frac{U^2}{\sigma^2} \right\} dU \\ &\quad = 1 - \frac{e^{-\rho}}{\sqrt{2\pi}} \beta(\rho) \end{aligned} \tag{72}$$

$$\begin{aligned}&E\{(U_i(\tau))^2\} \\ &\quad = E_s \sum_{m=1}^{K} \sum_{n=1}^{K} \dot{g}^2((m-n)T) - \frac{\sigma^2}{2} \ddot{g}(0) \end{aligned} \tag{73}$$

$$\begin{aligned}&E_{i,l} \\ &\quad = -\dot{g}^2((i-l)T) \left(\frac{e^{-\rho}}{\sqrt{\pi\sigma^2}} \int_{-\infty}^{+\infty} U \sinh\left(\frac{2\sqrt{E_s} U}{\sigma^2} \right) \right. \\ &\qquad \left. \times \exp\left\{ -\frac{U^2}{\sigma^2} \right\} dU \right)^2 \\ &\quad = -\rho \dot{g}^2((i-l)T). \end{aligned} \tag{74}$$

Finally, we obtain the closed-form expression for the stochastic CRLB of BPSK-modulated signals as follows:

$$\mathrm{CRLB_{BPSK}} = \frac{1}{4\rho}\left[\left(1-\frac{e^{-\rho}}{\sqrt{2\pi}}\beta(\rho)\right)\times\left(\rho\sum_{m=1}^{K}\sum_{n=1}^{K}\dot{g}^2((m-n)T)-\frac{K}{2}\ddot{g}(0)\right)-\rho\sum_{m=1}^{K}\sum_{n=1}^{K}\dot{g}((m-n)T)\right]^{-1} \quad (75)$$

where $\beta(\rho)$ is defined in (49). Now, consider a MSK-modulated signal. In order to find the derivative of (47) with respect to the time delay τ, we need to separate the cases where b_i is real or imaginary. To do so, we assume, without loss of generality, that K is an even number (i.e., $K = 2P$) and $a_0 = 1$. Using these assumptions, the log-likelihood function can be written as

$$L(\tau) = \sum_{i=1}^{P}\ln\left(\cosh\left(\frac{2\sqrt{E_s}}{\sigma^2}U_{2i-1}(\tau)\right)\right)+\ln\left(\cosh\left(\frac{2\sqrt{E_s}}{\sigma^2}V_{2i}(\tau)\right)\right) \quad (76)$$

where

$$U_i(\tau) = \int_{-\infty}^{+\infty}\Re\{\tilde{y}(t)\}h(t-iT-\tau)dt \quad (77)$$

and

$$V_i(\tau) = \int_{-\infty}^{+\infty}\Im\{\tilde{y}(t)\}h(t-iT-\tau)dt. \quad (78)$$

Then, the first derivative of (76) with respect to τ is given by

$$\frac{\partial L(\tau)}{\partial\tau} = \frac{2\sqrt{E_s}}{\sigma^2}\sum_{i=1}^{P}\left(\tanh\left(\frac{2\sqrt{E_s}}{\sigma^2}U_{2i-1}(\tau)\right)\dot{U}_{2i-1}(\tau)+\tanh\left(\frac{2\sqrt{E_s}}{\sigma^2}V_{2i}(\tau)\right)\dot{V}_{2i}(\tau)\right), \quad (79)$$

with $\dot{U}_{2i-1}(\tau)$ and $\dot{V}_{2i}(\tau)$ being the derivatives of $U_{2i-1}(\tau)$ and $V_{2i}(\tau)$ with respect to τ, respectively. Then, the first diagonal element of the FIM matrix is expressed as

$$\begin{aligned}[\boldsymbol{I}(\boldsymbol{\nu})]_{1;1} &= E\left\{\left(\frac{\partial L(\tau)}{\partial\tau}\right)^2\right\}\\ &= \frac{4E_s}{\sigma^4}\sum_{i=1}^{P}\sum_{l=1}^{P}E\left\{\tanh\left(\frac{2\sqrt{E_s}}{\sigma^2}U_{2i-1}(\tau)\right)\times\tanh\left(\frac{2\sqrt{E_s}}{\sigma^2}U_{2l-1}(\tau)\right)\dot{U}_{2i-1}(\tau)\dot{U}_{2l-1}(\tau)\right\}\\ &\quad+2\sum_{i=1}^{P}\sum_{l=1}^{P}E\left\{\tanh\left(\frac{2\sqrt{E_s}}{\sigma^2}U_{2i-1}(\tau)\right)\times\tanh\left(2\frac{\sqrt{E_s}}{\sigma^2}V_{2l}(\tau)\right)\times\dot{U}_{2i-1}(\tau)\dot{V}_{2l}(\tau)\right\}\\ &\quad+\sum_{i=1}^{P}\sum_{l=1}^{P}E\left\{\tanh\left(\frac{2\sqrt{E_s}}{\sigma^2}V_{2i}(\tau)\right)\times\tanh\left(\frac{2\sqrt{E_s}}{\sigma^2}V_{2l}(\tau)\right)\dot{V}_{2i}(\tau)\dot{V}_{2l}(\tau)\right]\Bigg\}. \end{aligned} \quad (80)$$

Note that (80) is equivalent to (31). Then for the same reasons, $[\boldsymbol{I}(\boldsymbol{\nu})]_{1;1}$ reduces simply to

$$[\boldsymbol{I}(\boldsymbol{\nu})]_{1;1} = \frac{8E_s}{\sigma^2}\sum_{i=1}^{P}\sum_{l=1}^{P}E\left\{\tanh\left(\frac{2\sqrt{\rho}}{\sigma^2}U_{2i-1}\right)=\tanh\left(\frac{2\sqrt{\rho}}{\sigma^2}U_{2l-1}\right)\dot{U}_{2i-1}\dot{U}_{2l-1}\right\} \quad (81)$$

which is similar to (70) in the case of BPSK modulation. Thus, we obtain the same expression for the stochastic CRLB in case of MSK and BPSK transmissions as given by (75).

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